Existence of weak solutions to some stationary Schrödinger equations with singular nonlinearity

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Abstract

We prove some existence (and sometimes also uniqueness) of weak solutions to some stationary equations associated to the complex Schrödinger operator under the presence of a singular nonlinear term. Among other new facts, with respect some previous results in the literature for such type of nonlinear potential terms, we include the case in which the spatial domain is possibly unbounded (something which is connected with some previous localization results by the authors), the presence of possible non-local terms at the equation, the case of boundary conditions different to the Dirichlet ones and, finally, the proof of the existence of solutions when the right-hand side term of the equation is beyond the usual L^2-space.

Contents

†The research of J.I. Díaz was partially supported by the project ref. MTM2011-26119 of the DGISPI (Spain) and the Research Group MOMAT (Ref. 910480) supported by UCM. He has received also support from the ITN *FIRST* of the Seventh Framework Program of the European Community's (grant agreement number 238702)

2010 Mathematics Subject Classification: 35Q55, (35A01, 35A02, 35B45, 35B65, 35J60)

Key Words: nonlinear Schrödinger equation, different boundary conditions, unbounded domains, non local terms, data in weighted spaces, existence, uniqueness, smoothness

1 Introduction

This paper is concerned by existence of weak solutions for two kinds of equations related to the complex Schrödinger operator,

$$-\Delta u + a|u|^{-(1-m)}u + bu = F, \text{ in } L^2(\Omega), \tag{1.1}$$

$$-\Delta u + a|u|^{-(1-m)}u + bu + cV^2u = F, \text{ in } L^2(\Omega), \tag{1.2}$$

with homogeneous Dirichlet boundary condition

$$u_{|\Gamma} = 0, \tag{1.3}$$

or homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu}_{|\Gamma} = 0, \tag{1.4}$$

where Ω is nonempty subset of $\mathbb{R}^N$ with boundary Γ, $0 < m < 1$, $(a,b,c) \in \mathbb{C}^3$ and $V \in L^\infty(\Omega;\mathbb{R})$ is a real potential. Here and in what follows, $\Delta = \sum\limits_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian in Ω.

In Bégout and Díaz [3], the authors study the spatial localization property compactness of the support of weak solutions of equation (1.1) (see Theorems 3.1, 3.5, 3.6, 4.1, 4.4 and 5.2). Existence, uniqueness and *a priori* bound are also established with the homogeneous Dirichlet boundary condition, $F \in L^p(\Omega)$ $(2 < p < \infty)$ and $(a,b) \in \mathbb{C}^2$ satisfying assumptions (2.15) below. In this paper, we give such existence and *a priori* bound results but for the weaker assumption $F \in L^2(\Omega)$ (Theorems 2.9 and 2.11) and also for some different hypotheses on $(a,b) \in \mathbb{C}^2$ (Theorems 2.4 and 2.5). Additionally, we consider homogeneous Neumann boundary condition (Theorems 2.9 and 2.11).

In Bégout and Díaz [1], spatial localization property for the partial differential equation (1.2) associated to self-similar solutions of the nonlinear Schrödinger equation

$$\mathrm{i}u_t + \Delta u = a|u|^{-(1-m)}u + f(t,x),$$

is studied.

In this paper, we prove existence of weak solutions with homogeneous Dirichlet or Neumann boundary

conditions (Theorems 2.6) and establish *a priori* bound (Theorem 2.7). For both equations (1.1) and (1.2) with any of both boundary conditions (1.3) or (1.4), we also show uniqueness (Theorem 2.12) and regularity results (Theorem 2.15), under suitable additional conditions. We send the reader to the long introduction of Bégout and Díaz [1] for many comments on the frameworks in which the equation arises (Quantum Mechanics, Nonlinear Optics and Hydrodynamics) and their connections with some other papers in the literature.

This paper is organized as follows. In the next section, we give results about existence, uniqueness, regularity and *a priori* bounds for equations (1.1) and (1.2), with boundary conditions (1.3) or (1.4). In Section 3, we give some very simple estimates which will be used to prove the results of this present section. Section 4, is devoted to the establishment of *a priori* bounds for the different truncated nonlinearities of equations studied in this paper. In Section 5, we prove the results given in Section 2. In Bégout and Díaz [3], localization property is studied for equation (1.1). The results we give require, sometimes, the same assumptions on $(a,b)\in\mathbb{C}^2$ as in Bégout and Díaz [3] but with a change of notation. Consequently, in Section 7 we present some planar representations of the assumptions on the complex parameters a and b which may help to understand this new notation (see also Comments 2.8 below for the motivation of this change). Finally, in Section 6 we will show the existence of solutions to equation (1.2) for data in a weighted subspace. Existence of weak solutions for equation (1.2) is used in Bégout and Díaz [1] while existence of weak solutions for equation (1.1) is used in Bégout and Díaz [2].

2 Main results and notations

Before stating our main results, we shall indicate here some of the notations used throughout. We write $\mathrm{i}^2=-1$. We denote by $\overline{z}$ the conjugate of the complex number z, by $\mathrm{Re}(z)$ its real part and by $\mathrm{Im}(z)$ its imaginary part. For $1\leqslant p\leqslant\infty$, p' is the conjugate of p defined by $\frac{1}{p}+\frac{1}{p'}=1$. The closure of a subset Ω of $\mathbb{R}^N$ is denoted by $\overline{\Omega}$ and its complement by $\Omega^\mathrm{c}=\mathbb{R}^N\setminus\Omega$. The notation $\omega\Subset\Omega$ means that the closure $\overline{\omega}\subset\Omega$ and that $\overline{\omega}$ is a compact subset of $\mathbb{R}^N$. For $x_0\in\mathbb{R}^N$ and $r>0$, we denote by $B(x_0,r)$ the open ball of $\mathbb{R}^N$ of center x_0 and radius r, by $\mathbb{S}(x_0,r)$ its boundary and by $\overline{B}(x_0,r)$ its closure. Unless if specified, any function belonging in a functional space $\big(L^p(\Omega)$, $W^{m,p}(\Omega)$, $H^1_0(\Omega)$, etc$\big)$ is supposed to be a complex-valued function $\big(L^p(\Omega;\mathbb{C})$, $W^{m,p}(\Omega;\mathbb{C})$, $H^1_0(\Omega;\mathbb{C})$, etc$\big)$. For a Banach space E, we denote by $E^\star$ its topological dual and by $\langle\,.\,,\,.\,\rangle_{E^\star,E}\in\mathbb{R}$ the $E^\star-E$ duality product. In particular, for any $T\in L^{p'}(\Omega)$ and $\varphi\in L^p(\Omega)$ with $1\leqslant p<\infty$, $\langle T,\varphi\rangle_{L^{p'}(\Omega),L^p(\Omega)}=\mathrm{Re}\int\limits_\Omega T(x)\overline{\varphi(x)}\mathrm{d}x$. We denote by $SO_N(\mathbb{R})$ the special orthogonal group of $\mathbb{R}^N$. As usual, we denote by C auxiliary

positive constants, and sometimes, for positive parameters $a_1,\dots,a_n$, write $C(a_1,\dots,a_n)$ to indicate that the constant C continuously depends only on $a_1,\dots,a_n$ (this convention also holds for constants which are not denoted by "C").

Definition 2.1. Let Ω be a nonempty open subset of $\mathbb{R}^N$, let $f\in C\big(H^1(\Omega);L^2(\Omega)\big)$ and let $F\in L^2(\Omega)$.

1) We say that u is a *weak solution* to
$$-\Delta u+f(u)=F,\ \text{ in } L^2(\Omega), \tag{2.1}$$
and
$$u_{|\Gamma}=0,$$
if $u\in H^1_0(\Omega)$ and if
$$\langle\nabla u,\nabla v\rangle_{L^2(\Omega),L^2(\Omega)}+\langle f(u),v\rangle_{L^2(\Omega),L^2(\Omega)}=\langle F,v\rangle_{L^2(\Omega),L^2(\Omega)}, \tag{2.2}$$
for any $v\in H^1_0(\Omega)$.

2) Assume that $\Gamma\neq\emptyset$ and Ω has a C^1 boundary. Let ν be the outward unit normal vector to Γ. We say that u is a *weak solution* to (2.1) and
$$\frac{\partial u}{\partial\nu}_{|\Gamma}=0,$$
if $u\in H^1(\Omega)$ and if u satisfies (2.2) for any $v\in H^1(\Omega)$.

In particular, for both kinds of boundary conditions, $\Delta u\in L^2(\Omega)$. It follows then that $u\in H^2_{\mathrm{loc}}(\Omega)$ and so (2.1) takes sense almost everywhere in Ω (see, for instance, Proposition 4.1.2, p.101-102 in Cazenave [7]).

3) In a general way, if one has merely $f\in C\big(L^p_{\mathrm{loc}}(\Omega);L^1_{\mathrm{loc}}(\Omega)\big)$ and $F\in L^1_{\mathrm{loc}}(\Omega)$, for some $1\leqslant p\leqslant\infty$, then we say that u is a *local very weak solution* to
$$-\Delta u+f(u)=F,\ \text{ in } \mathscr{D}'(\Omega),$$
if $u\in L^p_{\mathrm{loc}}(\Omega)$ and if
$$\langle -u,\Delta\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}+\langle f(u),\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}=\langle F,\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)},$$
for any $\varphi\in\mathscr{D}(\Omega)$. If, in addition, $f\in C\big(L^p(\Omega);L^1(\Omega)\big)$ and $u\in L^p(\Omega)$ then u is said to be a *global very weak solution.*

Remark 2.2. Here are some comments about Definition 2.1.

1) More generally, we may introduce some special issues classes of weak solutions: let $\mathscr{D}(\Omega) \hookrightarrow E \subset L^1_{\mathrm{loc}}(\Omega)$ be a Banach space with dense embedding. Let
$$f \in C\big(E;E^\star\big) \cap C\big(H^1_{\mathrm{loc}}(\Omega);L^1_{\mathrm{loc}}(\Omega)\big),$$
and let $F \in L^2(\Omega)$. Then u is said to be a weak solution to
$$-\Delta u + f(u) = F, \text{ in } E^\star + L^2(\Omega), \tag{2.3}$$
and (1.3) $\big($respectively, to (2.3) and (1.4)$\big)$ if
$$u \in H^1_0(\Omega) \cap E \quad \text{(respectively, if } u \in H^1(\Omega) \cap E\text{)},$$
and if
$$\langle \nabla u, \nabla v\rangle_{L^2(\Omega),L^2(\Omega)} + \langle f(u), v\rangle_{E^\star,E} = \langle F, v\rangle_{L^2(\Omega),L^2(\Omega)}, \tag{2.4}$$
for any $v \in H^1_0(\Omega) \cap E$ (respectively, for any $v \in H^1(\Omega) \cap E$). In particular, for both kinds of boundary conditions, $\Delta u \in E^\star + L^2(\Omega)$ and so (2.3) takes sense in $E^\star + L^2(\Omega)$. It follows that if $E^\star \subset L^1_{\mathrm{loc}}(\Omega)$ then (2.3) takes sense almost everywhere in Ω. Note that since $f \in C\big(H^1_{\mathrm{loc}}(\Omega);L^1_{\mathrm{loc}}(\Omega)\big)$ and $\Delta u \in \mathscr{D}'(\Omega)$ (with help of the dense embedding $\mathscr{D}(\Omega) \hookrightarrow E$), any weak solution in $E^\star + L^2(\Omega)$ is also a solution in $\mathscr{D}'(\Omega)$.

2) Assume that Ω is bounded and has a $C^{0,1}$ boundary. We recall that if $u \in H^1(\Omega)$ then boundary condition $u_{|\Gamma} = 0$ makes sense in the sense of the trace $\gamma(u) = 0$ and that $u \in H^1_0(\Omega)$ if and only if $\gamma(u) = 0$. If furthermore Ω has a C^1 boundary and if $u \in C(\overline{\Omega}) \cap H^1_0(\Omega)$ then for any $x \in \Gamma$, $u(x) = 0$ (Theorem 9.17, p.288, in Brezis [4]). Finally, if $u \notin C(\overline{\Omega})$ and Ω has not a $C^{0,1}$ boundary, the condition $u_{|\Gamma} = 0$ does not take sense and, in this case, has to be understood as $u \in H^1_0(\Omega)$.

3) Assume that Ω is bounded and has a $C^{1,1}$ boundary. Let ν be the outward unit normal vector to Γ and let $u \in H^1(\Omega)$ be any weak solution to (2.1) with homogeneous Neumann boundary condition (1.4). Then $u \in H^2(\Omega)$ and boundary condition $\frac{\partial u}{\partial \nu}_{|\Gamma} = 0$ makes sense in the sense of the trace $\gamma\big(\nabla u.\nu\big) = 0$. If, in addition, $u \in C^1(\overline{\Omega})$ then obviously for any $x \in \Gamma$, $\frac{\partial u}{\partial \nu}(x) = 0$. Indeed, since $u \in H^1(\Omega)$, $\Delta u \in L^2(\Omega)$ and (2.1) takes sense almost everywhere

in Ω, we have $\gamma\left(\frac{\partial u}{\partial\nu}\right)\in H^{-\frac{1}{2}}(\Gamma)$ and by Green's formula,

$$\operatorname{Re}\int\limits_\Omega \nabla u(x).\overline{\nabla v(x)}\mathrm{d}x-\left\langle\gamma\left(\frac{\partial u}{\partial\nu}\right),\gamma(v)\right\rangle_{H^{-\frac{1}{2}}(\Gamma),H^{\frac{1}{2}}(\Gamma)}$$
$$+\operatorname{Re}\int\limits_\Omega f\big(u(x)\big)\overline{v(x)}\mathrm{d}x=\operatorname{Re}\int\limits_\Omega F(x)\overline{v(x)}\mathrm{d}x, \quad (2.5)$$

for any $v\in H^1(\Omega)$ (see Lemma 4.1, Theorem 4.2 and Corollary 4.1, p.155, in Lions and Magenes [16] and (1,5,3,10) in Grisvard [12], p.62). By Definition 2.1, this implies that

$$\left\langle\gamma\left(\frac{\partial u}{\partial\nu}\right),\gamma(v)\right\rangle_{H^{-\frac{1}{2}}(\Gamma),H^{\frac{1}{2}}(\Gamma)}=0, \tag{2.6}$$

for any $v\in H^1(\Omega)$. Let $w\in H^{\frac{1}{2}}(\Gamma)$. Let $v\in H^1(\Omega)$ be such that $\gamma(v)=w$ (Theorem 1.5.1.3, p.38, in Grisvard [12]). We then deduce from (2.6) that,

$$\forall w\in H^{\frac{1}{2}}(\Gamma),\ \left\langle\gamma\left(\frac{\partial u}{\partial\nu}\right),w\right\rangle_{H^{-\frac{1}{2}}(\Gamma),H^{\frac{1}{2}}(\Gamma)}=0,$$

and so $\gamma\left(\frac{\partial u}{\partial\nu}\right)=0$. But also $u\in L^2(\Omega)$ and $\Delta u\in L^2(\Omega)$. It follows that $u\in H^2(\Omega)$ (Proposition 2.5.2.3, p.131, in Grisvard [12]). Hence the result.

Example 2.3. Here, we give some examples which fall into the scope of 1) of Remark 2.2.

1) Let $\Omega\subseteq\mathbb{R}^N$ be a nonempty open subset, let $(a,b,c)\in\mathbb{C}^3$, let $0<m\leqslant 1$, let $V\in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R})$ and set

$$\Sigma=\left\{u\in L^2(\Omega);Vu\in L^2(\Omega)\right\},\ (u|v)_\Sigma=(u|v)_{L^2}+\operatorname{Re}\int\limits_\Omega V^2u\overline{v}\mathrm{d}x,\ \|u\|_\Sigma^2=(v|v)_\Sigma,$$

where $(\,.\,|\,.\,)_{L2}$ denotes the scalar product in $L^2(\Omega)$. We obviously have that $\big(\Sigma,(\,.\,|\,.\,)_\Sigma,\|\,.\,\|_\Sigma\big)$ is a Hilbert space and that $\mathscr{D}(\Omega)\hookrightarrow\Sigma$ with dense embedding. For any $u\in\Sigma$, let us define

$$g(u)=V^2u.$$

Let $u\in\Sigma$ and let $(u_n)_{n\in\mathbb{N}}\subset\Sigma$ be such that $u_n\xrightarrow[n\to\infty]{\Sigma}u$. It follows from the dense embedding $\mathscr{D}(\Omega)\hookrightarrow\Sigma$ and Cauchy-Schwarz's inequality that,

$$\|g(u)-g(u_n)\|_{\Sigma^\star}=\sup_{\begin{cases}v\in\mathscr{D}(\Omega)\\ \|v\|_\Sigma=1\end{cases}}\big\langle g(u)-g(u_n),v\big\rangle_{\Sigma^\star,\Sigma}$$
$$=\sup_{\begin{cases}v\in\mathscr{D}(\Omega)\\ \|v\|_\Sigma=1\end{cases}}\big\langle g(u)-g(u_n),v\big\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}=\sup_{\begin{cases}v\in\mathscr{D}(\Omega)\\ \|v\|_\Sigma=1\end{cases}}\operatorname{Re}\int\limits_\Omega V^2(x)\big(u(x)-u_n(x)\big)\overline{v(x)}\mathrm{d}x$$
$$\leqslant\big\|V\big(u-u_n\big)\big\|_{L^2(\Omega)}\sup_{\begin{cases}v\in\mathscr{D}(\Omega)\\ \|v\|_\Sigma=1\end{cases}}\big\|Vv\big\|_{L^2(\Omega)}\leqslant\|u-u_n\|_\Sigma\xrightarrow{n\to\infty}0.$$

So that, $g \in C\big(\Sigma;\Sigma^\star\big)$. In addition, since $\mathscr{D}(\Omega)$ is dense in both $L^{m+1}(\Omega)$ and Σ, it follows that

$$\big(\Sigma \cap L^{m+1}(\Omega)\big)^\star = \Sigma^\star + L^{\frac{m+1}{m}}(\Omega),$$

(see, for instance, Proposition 1.1.3, p.3, in Cazenave [6]). Now consider $E = \Sigma \cap L^{m+1}(\Omega)$ and $f(u) = a|u|^{-(1-m)}u + bu + cVu$. From what precedes, we have that $E^\star = \Sigma^\star + L^{\frac{m+1}{m}}(\Omega)$ and

$$f \in C\big(E;E^\star\big) \cap C\big(H^1_{\mathrm{loc}}(\Omega);L^1_{\mathrm{loc}}(\Omega)\big).$$

Applying 1) of Remark 2.2, we obtain that for any $F \in L^2(\Omega)$, u is a weak solution to

$$-\Delta u + a|u|^{-(1-m)}u + bu + cV^2u = F, \text{ in } \Sigma^\star + L^{\frac{m+1}{m}}(\Omega), \tag{2.7}$$

and (1.3) $\big($respectively, to (2.7) and (1.4)$\big)$ if

$$u \in H^1_0(\Omega) \cap L^{m+1}(\Omega) \cap \Sigma \quad \big(\text{respectively, if } u \in H^1(\Omega) \cap L^{m+1}(\Omega) \cap \Sigma\big),$$

and

$$\begin{aligned}\langle \nabla u, \nabla v\rangle_{L^2(\Omega),L^2(\Omega)} & \\ + \langle a|u|^{-(1-m)}u + bu + cV^2u, v\rangle_{\Sigma^\star + L^{\frac{m+1}{m}}(\Omega),\Sigma\cap L^{m+1}(\Omega)} & \\ = \langle F, v\rangle_{L^2(\Omega),L^2(\Omega)}, &\end{aligned} \tag{2.8}$$

for any $v \in H^1_0(\Omega) \cap L^{m+1}(\Omega) \cap \Sigma$ $\big($respectively, for any $v \in H^1(\Omega) \cap L^{m+1}(\Omega) \cap \Sigma\big)$. In particular, for both kinds of boundary conditions, $\Delta u \in \big(\Sigma^\star + L^{\frac{m+1}{m}}(\Omega)\big) \cap L^1_{\mathrm{loc}}(\Omega)$ and so (2.7) takes sense in $\big(\Sigma^\star + L^{\frac{m+1}{m}}(\Omega)\big) \cap L^1_{\mathrm{loc}}(\Omega)$ and almost everywhere in Ω.

2) Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $a \in \mathbb{C}$, let $0 < m \leqslant 1$, let $V \in L^\infty(\Omega;\mathbb{C})$ and set

$$f(u) = a|u|^{-(1-m)}u + Vu.$$

Since $\mathscr{D}(\Omega)$ is dense in both $L^{m+1}(\Omega)$ and $L^2(\Omega)$, it follows that

$$\big(L^2(\Omega) \cap L^{m+1}(\Omega)\big)^\star = L^2(\Omega) + L^{\frac{m+1}{m}}(\Omega),$$

(see, for instance, Proposition 1.1.3, p.3, in Cazenave [6]). One easily checks that

$$f \in C\big(L^2(\Omega) \cap L^{m+1}(\Omega); L^2(\Omega) + L^{\frac{m+1}{m}}(\Omega)\big) \cap C\big(H^1_{\mathrm{loc}}(\Omega);L^1_{\mathrm{loc}}(\Omega)\big).$$

Applying 1) of Remark 2.2, we obtain that for any $F\in L^2(\Omega)$, u is a weak solution to

$$-\Delta u+a|u|^{-(1-m)}u+Vu=F, \text{ in } L^2(\Omega)+L^{\frac{m+1}{m}}(\Omega), \tag{2.9}$$

and (1.3) $\big($respectively to (2.9) and (1.4)$\big)$ if

$$u\in H^1_0(\Omega)\cap L^{m+1}(\Omega) \quad \big(\text{respectively, if } u\in H^1(\Omega)\cap L^{m+1}(\Omega)\big),$$

and if

$$\begin{aligned}\langle\nabla u,\nabla v\rangle_{L^2(\Omega),L^2(\Omega)}&\\ +\,\langle a|u|^{-(1-m)}u+Vu,v\rangle_{L^2(\Omega)+L^{\frac{m+1}{m}}(\Omega),L^2(\Omega)\cap L^{m+1}(\Omega)}&\\ =\langle F,v\rangle_{L^2(\Omega),L^2(\Omega)},&\end{aligned} \tag{2.10}$$

for any $v\in H^1_0(\Omega)\cap L^{m+1}(\Omega)$ $\big($respectively, for any $v\in H^1(\Omega)\cap L^{m+1}(\Omega)\big)$. In particular, for both kinds of boundary conditions, $\Delta u\in L^2(\Omega)+L^{\frac{m+1}{m}}(\Omega)$ and so (2.9) takes sense in $L^2(\Omega)+L^{\frac{m+1}{m}}(\Omega)$ and almost everywhere in Ω.

3) **Hartree-Fock type equations.** Let $V\in L^p(\mathbb{R}^N;\mathbb{R})+L^\infty(\mathbb{R}^N;\mathbb{R})$, with $\min\left\{1,\frac{N}{2}\right\}<p<\infty$ and let $W\in L^q(\mathbb{R}^N;\mathbb{R})+L^\infty(\mathbb{R}^N;\mathbb{R})$, with $\min\left\{1,\frac{N}{4}\right\}<q<\infty$. Set $r=\frac{2p}{p-1}$, $s=\frac{4q}{q-1}$,

$$E=L^2(\mathbb{R}^N)\cap L^4(\mathbb{R}^N)\cap L^r(\mathbb{R}^N)\cap L^s(\mathbb{R}^N),$$
$$f(u)=Vu+(W\star|u|^2)u,$$

for any $u\in H^1(\mathbb{R}^N)$. Then $H^1(\mathbb{R}^N)\hookrightarrow E$ with dense embedding and by density of $\mathscr{D}(\mathbb{R}^N)$ in spaces $L^m(\mathbb{R}^N)$, for any $m\in[1,\infty)$, we have

$$E^\star=L^2(\mathbb{R}^N)+L^{\frac{4}{3}}(\mathbb{R}^N)+L^{r'}(\mathbb{R}^N)+L^{s'}(\mathbb{R}^N),$$
$$f\in C\big(E;E^\star\big),$$
$$f\in C\big(H^1(\mathbb{R}^N);H^{-1}(\mathbb{R}^N)\big).$$

See Cazenave [6], Proposition 1.1.3, p.3, Proposition 3.2.2, p.58-59, Remark 3.2.3, p.59, Proposition 3.2.9, p.62, Remark 3.2.10, p.63 and Example 3.2.11, p.63.

Let $0<m\leqslant1$ and let $z\in\mathbb{C}\setminus\{0\}$. Since $\big||z|^{-(1-m)}z\big|=|z|^m$, it is understood in (2.7) and (2.9) of Example 2.3 that $\big||z|^{-(1-m)}z\big|=0$ when $z=0$.

We start now with the statements of the main results of this paper.

Theorem 2.4 (**Existence**). *Let Ω a nonempty open subset of $\mathbb{R}^N$ be such that $|\Omega| < \infty$ and assume $0 < m < 1$, $(a,b) \in \mathbb{C}^2$ and $F \in L^2(\Omega)$. If $\mathrm{Re}(b) < 0$ then assume further that $\mathrm{Im}(b) \neq 0$ or $-\frac{1}{C_\mathrm{P}^2} < \mathrm{Re}(b)$, where C_P is the constant in (3.2) of Lemma 3.1 below. Then there exists at least one weak solution $u \in H_0^1(\Omega)$ to (1.1) and (1.3). In addition, any weak solution belongs to $H^2_{\mathrm{loc}}(\Omega)$.*

Symmetry Property. *If furthermore, for any $\mathcal{R} \in SO_N(\mathbb{R})$, $\mathcal{R}\Omega = \Omega$ and if F is spherically symmetric then there exists a spherically symmetric weak solution $u \in H_0^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$ of (1.1) and (1.3). For $N = 1$, this means that if F is an even (respectively, an odd) function then u is also an even (respectively, an odd) function.*

Theorem 2.5 (***A priori*** **bound**). *Let Ω a nonempty open subset of $\mathbb{R}^N$ be such that $|\Omega| < \infty$ and assume $0 < m < 1$, $(a,b) \in \mathbb{C}^2$ and $F \in L^2(\Omega)$. If $\mathrm{Re}(b) < 0$ then assume further that $\mathrm{Im}(b) \neq 0$ or $-\frac{1}{C_\mathrm{P}^2} < \mathrm{Re}(b)$, where C_P is the constant in (3.2) of Lemma 3.1 below. Let $u \in H_0^1(\Omega)$ be any weak solution to (1.1) and (1.3). Then we have the following estimate.*

$$\|u\|_{H_0^1(\Omega)} \leqslant C,$$

where $C = C(\|F\|_{L^2(\Omega)}, |\Omega|, |a|, |b|, N, m)$.

Theorem 2.6 (**Existence**). *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset and assume $V \in L^\infty(\Omega;\mathbb{R})$, $0 < m < 1$, $(a,b,c) \in \mathbb{C}^3$ is such that $\mathrm{Im}(a) \leqslant 0$, $\mathrm{Im}(b) < 0$ and $\mathrm{Im}(c) \leqslant 0$. If $\mathrm{Re}(a) \leqslant 0$ then assume further that $\mathrm{Im}(a) < 0$. Then we have the following result.*

1) *For any $F \in L^2(\Omega)$, there exists at least one weak solution $u \in H_0^1(\Omega) \cap L^{m+1}(\Omega)$ to (1.2) and (1.3). Furthermore, any weak solution belongs to $H^2_{\mathrm{loc}}(\Omega)$.*
2) *If we assume furthermore that Ω is bounded with a C^1 boundary then the conclusion 1) still holds true with $u \in H^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$ and the homogeneous Neumann boundary condition (1.4) instead of $u \in H_0^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$ and the homogeneous Dirichlet boundary condition (1.3).*

Symmetry Property. *If furthermore, for any $\mathcal{R} \in SO_N(\mathbb{R})$, $\mathcal{R}\Omega = \Omega$ and if $F \in L^2(\Omega)$ is spherically symmetric then there exists a spherically symmetric weak solution $u \in H^1(\Omega) \cap L^{m+1}(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$ of (1.2) satisfying the desired boundary condition (1.3) or (1.4)[1]. For $N = 1$, this means that if F is an even (respectively, an odd) function then u is also an even (respectively, an odd) function.*

Theorem 2.7 (***A priori*** **bound**). *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $V \in L^\infty(\Omega;\mathbb{R})$, let $0 < m < 1$, let $(a,b,c) \in \mathbb{C}^3$ be such that $\mathrm{Im}(a) \leqslant 0$, $\mathrm{Im}(b) < 0$ and $\mathrm{Im}(c) \leqslant 0$. If $\mathrm{Re}(a) \leqslant 0$ then*

[1] for which we additionally assume that Ω has a C^1 boundary.

assume further that $\mathrm{Im}(a)<0$. *Let* $F\in L^2(\Omega)$ *and let* $u\in H^1(\Omega)$ *be any weak solution to* (1.2) *with boundary condition* (1.3) *or* (1.4)[1]. *Then we have the following estimate.*

$$\|u\|_{H^1(\Omega)}^2+\|u\|_{L^{m+1}(\Omega)}^{m+1}\leqslant M\big(\|V\|_{L^\infty(\Omega)}^4+1\big)\|F\|_{L^2(\Omega)}^2,$$

where $M=M(|a|,|b|,|c|)$.

Comments 2.8. In the context of the paper of Bégout and Díaz [3], we can establish an existence result with the homogeneous Neumann boundary condition (instead of the homogeneous Dirichlet condition) and $F\in L^2(\Omega)$ $\big($instead of $F\in L^{\frac{m+1}{m}}(\Omega)\big)$. In Bégout and Díaz [3], we introduced the set,

$$\widetilde{\mathbb{A}}=\mathbb{C}\setminus\big\{z\in\mathbb{C};\mathrm{Re}(z)=0\text{ and }\mathrm{Im}(z)\leqslant0\big\},$$

and assumed that $(\widetilde{a},\widetilde{b})\in\mathbb{C}^2$ satisfies,

$$(\widetilde{a},\widetilde{b})\in\widetilde{\mathbb{A}}\times\widetilde{\mathbb{A}}\quad\text{and}\quad\begin{cases}\mathrm{Re}(\widetilde{a})\mathrm{Re}(\widetilde{b})\geqslant0,\\ \text{or}\\ \mathrm{Re}(\widetilde{a})\mathrm{Re}(\widetilde{b})<0\ \text{ and }\ \mathrm{Im}(\widetilde{b})>\dfrac{\mathrm{Re}(\widetilde{b})}{\mathrm{Re}(\widetilde{a})}\mathrm{Im}(\widetilde{a}),\end{cases}\tag{2.11}$$

with possibly $\widetilde{b}=0$, and we worked with

$$-\mathrm{i}\Delta u+\widetilde{a}|u|^{-(1-m)}u+\widetilde{b}u=\widetilde{F}.$$

Nevertheless, to maintain a closer notation to many applied works in the literature (see, e.g., the introduction of Bégout and Díaz [1]), we do not work any more with this equation but with,

$$-\Delta u+a|u|^{-(1-m)}u+bu=F,$$

and $b\neq0$. This means that we chose, $\widetilde{a}=\mathrm{i}a$, $\widetilde{b}=\mathrm{i}b$ and $\widetilde{F}=\mathrm{i}F$. Then assumptions on (a,b) are changed by the fact that,

$$\mathrm{Re}(a)=\mathrm{Re}(-\mathrm{i}\widetilde{a})=\mathrm{Im}(\widetilde{a}),\tag{2.12}$$

$$\mathrm{Im}(b)=\mathrm{Im}(-\mathrm{i}\widetilde{b})=-\mathrm{Re}(\widetilde{b}).\tag{2.13}$$

It follows that the set $\widetilde{\mathbb{A}}$ and (2.11) become,

$$\mathbb{A}=\mathbb{C}\setminus\big\{z\in\mathbb{C};\mathrm{Re}(z)\leqslant0\text{ and }\mathrm{Im}(z)=0\big\},\tag{2.14}$$

$$(a,b)\in\mathbb{A}\times\mathbb{A}\quad\text{and}\quad\begin{cases}\mathrm{Im}(a)\mathrm{Im}(b)\geqslant0,\\ \text{or}\\ \mathrm{Im}(a)\mathrm{Im}(b)<0\ \text{ and }\ \mathrm{Re}(b)>\dfrac{\mathrm{Im}(b)}{\mathrm{Im}(a)}\mathrm{Re}(a).\end{cases}\tag{2.15}$$

Obviously,

$$\Big((\widetilde{a},\widetilde{b})\in\widetilde{\mathbb{A}}\times\widetilde{\mathbb{A}} \text{ satisfies } (2.11)\Big) \iff \Big((a,b)\in\mathbb{A}\times\mathbb{A} \text{ satisfies } (2.15)\Big).$$

Assumptions (2.15) are made to prove the existence and the localization property of weak solutions to equation (1.1). A geometric interpretation of (2.15) is given in Section 7 (as in Section 6 of Bégout and Díaz [3]). Now, we give some results about equation (1.1) when $(a,b)\in\mathbb{A}\times\mathbb{A}$ satisfies (2.15).

Theorem 2.9 (**Existence**). *Let Ω be a nonempty open subset of $\mathbb{R}^N$, let $0<m<1$ and let $(a,b)\in\mathbb{A}^2$ satisfies (2.15)[2].*

1) *For any $F\in L^2(\Omega)$, there exists at least one weak solution $u\in H^1_0(\Omega)\cap L^{m+1}(\Omega)$ to (1.1) and (1.3). Furthermore, any weak solution belongs to $H^2_{\mathrm{loc}}(\Omega)$.*
2) *If we assume furthermore that Ω is bounded with a C^1 boundary then the conclusion* 1) *still holds true with $u\in H^1(\Omega)\cap H^2_{\mathrm{loc}}(\Omega)$ and the homogeneous Neumann boundary condition (1.4) instead of $u\in H^1_0(\Omega\cap H^2_{\mathrm{loc}}(\Omega)$ and the homogeneous Dirichlet boundary condition (1.3).*

In addition, the Symmetry Property of Theorem 2.6 holds true for equation (1.1).

Remark 2.10. Here are some comments about Theorems 2.4, 2.6 and 2.9.

1) Assume F is spherically symmetric. Since we do not know, in general, if we have uniqueness of the weak solution, we are not able to show that any weak solution is radially symmetric.
2) Uniqueness for equation (1.1) holds under assumption $a\neq0$, $\mathrm{Re}(a)\geqslant0$ and $\mathrm{Re}(a\overline{b})\geqslant0$. Geometrically, this means that $\overrightarrow{a}.\overrightarrow{b}\geqslant0$ or, equivalently, $\left|\angle(\overrightarrow{a},\overrightarrow{b})\right|\leqslant\dfrac{\pi}{2}$rad. For more details, see Section 7. Extension is given in Theorem 2.12 below.

Theorem 2.11 (***A priori*** **bound**). *Let $\Omega\subseteq\mathbb{R}^N$ be a nonempty open subset of $\mathbb{R}^N$, let $0<m<1$ and let $(a,b)\in\mathbb{A}^2$ satisfies (2.15)[2]. Let $F\in L^2(\Omega)$ and let $u\in H^1(\Omega)\cap L^{m+1}(\Omega)$ be any weak solution[3] to*

$$-\Delta u+a|u|^{-(1-m)}u+bu=F, \text{ in } L^2(\Omega)+L^{\frac{m+1}{m}}(\Omega), \tag{2.16}$$

with boundary condition (1.3) or (1.4)[1]. Then we have the following estimate.

$$\|u\|^2_{H^1(\Omega)}+\|u\|^{m+1}_{L^{m+1}(\Omega)}\leqslant M\|F\|^2_{L^2(\Omega)},$$

where $M=M(|a|,|b|)$.

[2] See Comments 2.8 and Section 7.
[3] See 2) of Example 2.3.

Theorem 2.12 (**Uniqueness**)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $V \in L^\infty_{\text{loc}}(\Omega;\mathbb{R})$, let $0 < m < 1$ and let $(a,b,c) \in \mathbb{C}^3$ satisfies one of the three following conditions.*

1) $a \neq 0$, $\operatorname{Re}(a) \geqslant 0$, $\operatorname{Re}(a\overline{b}) \geqslant 0$ *and* $\operatorname{Re}(a\overline{c}) \geqslant 0$.
2) $b \neq 0$, $\operatorname{Re}(b) \geqslant 0$, $a = kb$, *for some* $k \geqslant 0$ *and* $\operatorname{Re}(b\overline{c}) \geqslant 0$.
3) $c \neq 0$, $\operatorname{Re}(c) \geqslant 0$, $a = kc$, *for some* $k > 0$ *and* $\operatorname{Re}(b\overline{c}) \geqslant 0$.

Let $F \in L^1_{\text{loc}}(\Omega)$. If there exist two weak solutions $u_1, u_2 \in H^1(\Omega) \cap L^{m+1}(\Omega)$ to equation (1.2) with the same boundary condition (1.3) or (1.4)[1] *such that $Vu_1, Vu_2 \in L^2(\Omega)$ then $u_1 = u_2$.*

Remark 2.13. Here are some comments about Theorem 2.12.

1) In Theorem 5.2 in Bégout and Díaz [3], uniqueness for equation
$$-\mathrm{i}\Delta u + \widetilde{a}|u|^{-(1-m)}u + \widetilde{b}u = \widetilde{F},$$
holds if $\widetilde{a} \neq 0$, $\operatorname{Im}(\widetilde{a}) \geqslant 0$ and $\operatorname{Re}(\widetilde{a}\overline{\widetilde{b}}) \geqslant 0$. By (2.12)–(2.13), those assumptions are equivalent to 1) of Theorem 2.12 above for equation (1.1) (of course, $c = 0$). It follows that Theorem 2.12 above extends Theorem 5.2 of Bégout and Díaz [3].

2) In 2) of the above theorem, if we want to make an analogy with 1), assumption $a = kb$, for some $k \geqslant 0$ has to be replaced with $\operatorname{Re}(a\overline{b}) \geqslant 0$ and $\operatorname{Im}(a\overline{b}) = 0$. But,
$$\Big(\operatorname{Re}(a\overline{b}) \geqslant 0 \text{ and } \operatorname{Im}(a\overline{b}) = 0\Big) \iff \Big(\exists k \geqslant 0/a = kb\Big).$$
In the same way,
$$\Big(\operatorname{Re}(a\overline{c}) > \text{ and } \operatorname{Im}(a\overline{c}) = 0\Big) \iff \Big(\exists k > 0/a = kc\Big).$$

3) Note that if $|\Omega| < \infty$ then $H^1(\Omega) \cap L^{m+1}(\Omega) = H^1(\Omega)$.

Remark 2.14. In the case of real weak solutions (with $F \equiv 0$ and $(a,b,c) \in \mathbb{R} \times \mathbb{R} \times \{0\}$), it is well-known that if $b < 0$ then it may appear multiplicity of weak solutions (once $m \in (0,1)$ and $a > 0$). For more details, see Theorem 1 in Díaz and Hernández [8].

Theorem 2.15 (**Regularity**)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $V \in L^r_{\text{loc}}(\Omega;\mathbb{C})$, for any $1 < r < \infty$, let $0 < m < 1$, let $(a,b) \in \mathbb{C}^2$, let $F \in L^1_{\text{loc}}(\Omega)$, let $1 < q < \infty$ and let $u \in L^q_{\text{loc}}(\Omega)$ be any local very weak solution to*
$$-\Delta u + a|u|^{-(1-m)}u + Vu = F, \text{ in } \mathscr{D}'(\Omega). \tag{2.17}$$
Let $q \leqslant p < \infty$ and let $\alpha \in (0,m]$.

1) *If* $F \in L^p_{\mathrm{loc}}(\Omega)$ *then* $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$. *If* $(F,V) \in C^{0,\alpha}_{\mathrm{loc}}(\Omega) \times C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ *then* $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$.

2) *Assume further that* Ω *is bounded with a* $C^{1,1}$ *boundary,* $F \in L^p(\Omega)$, $V \in L^r(\Omega;\mathbb{C})$, *for any* $1<r<\infty$, $u \in L^q(\Omega)$ *is a global very weak solution and* $u_{|\Gamma} = 0$ *in the sense of the trace. Then* $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$. *If* $(F,V) \in C^{0,\alpha}(\overline{\Omega}) \times C^{0,\alpha}(\overline{\Omega})$ *then* $u \in C^{2,\alpha}(\overline{\Omega}) \cap C_0(\Omega)$.

3) *Assume further that* Ω *is bounded with a* $C^{1,1}$ *boundary,* $F \in L^p(\Omega)$, $V \in L^r(\Omega;\mathbb{C})$, *for any* $1<r<\infty$, $u \in L^q(\Omega)$ *is a global very weak and* $\frac{\partial u}{\partial \nu}_{|\Gamma} = 0$ *in the sense of the trace, where* ν *denotes the outward unit normal vector to* Γ. *Then* $u \in W^{2,p}(\Omega)$. *If* $(F,V) \in C^{0,\alpha}(\overline{\Omega}) \times C^{0,\alpha}(\overline{\Omega})$ *then* $u \in C^{2,\alpha}(\overline{\Omega})$ *and for any* $x \in \Gamma$, $\frac{\partial u}{\partial \nu}(x) = 0$.

Here and in what follows, for $0 < \alpha \leqslant 1$ and $k \in \mathbb{N}_0 \overset{\text{def}}{=} \mathbb{N} \cup \{0\}$,

$$C^{k,\alpha}_{\mathrm{loc}}(\Omega) = \left\{ u \in C^k(\Omega;\mathbb{C}); \forall \omega \Subset \Omega,\ \sum_{|\beta|=k} H^\alpha_\omega(D^\beta u) < +\infty \right\},$$

where $H^\alpha_\omega(u) = \sup\limits_{\left\{\substack{(x,y)\in\omega^2 \\ x\neq y}\right.} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$ and

$$C^{k,\alpha}(\overline{\Omega}) = \Big\{ u \in C^k(\overline{\Omega};\mathbb{C}); \sum_{|\beta|=k} H^\alpha_\Omega(D^\beta u) < +\infty \Big\}.$$

Of course, $C(\Omega)$ or $C^0(\Omega)$ is the space of continuous functions from Ω to $\mathbb{C}$ and for $k \in \mathbb{N}$, $C^k(\Omega)$ is the space of functions lying in $C(\Omega;\mathbb{C})$ and having all derivatives of order lesser or equal than k belonging to $C(\Omega;\mathbb{C})$. Finally,

$$C_0(\overline{\Omega}) = \big\{ u \in C(\overline{\Omega};\mathbb{C}); \forall x \in \Gamma,\ u(x) = 0 \big\},$$

when Ω is bounded.

Remark 2.16. Let Ω be a nonempty bounded open subset of $\mathbb{R}^N$ with a $C^{1,1}$ boundary, let $V \in \bigcap\limits_{1<r<\infty} L^r(\Omega;\mathbb{C})$, let $0<m<1$, let $(a,b) \in \mathbb{C}^2$, let $1 < q \leqslant p < \infty$, let $F \in L^p(\Omega)$ and let $u \in L^q(\Omega)$ be any global very weak solution to (2.17). Let $T : u \longrightarrow \big\{ \gamma(u), \gamma\left(\frac{\partial u}{\partial \nu}\right) \big\}$ be the trace function defined on $\mathscr{D}(\overline{\Omega})$ and let $D_q(\Delta) = \big\{ u \in L^q(\Omega); \Delta u \in L^q(\Omega) \big\}$. By density of $\mathscr{D}(\overline{\Omega})$ in $D_q(\Delta)$, T has a linear and continuous extension from $D_q(\Delta)$ into $W^{-\frac{1}{q},q}(\Gamma) \times W^{-1-\frac{1}{q},q}(\Gamma)$ (Hörmander [13], Theorem 2 p.503; Lions and Magenes [16], Lemma 2.2 and Theorem 2.1 p.147; Lions and Magenes [17], Propositions 9.1, Proposition 9.2 and Theorem 9.1 p.82; Grisvard [12], p.54). Since $u \in L^q(\Omega)$, it follows from equation (2.17) and Hölder's inequality that $u \in D_q(\Delta)$. Then "$u_{|\Gamma} = 0$ in the sense of the trace" and "$\frac{\partial u}{\partial \nu}_{|\Gamma} = 0$ in the sense of the trace" make sense and means that $\gamma(u) = 0$ and $\gamma\left(\frac{\partial u}{\partial \nu}\right) = 0$, respectively.

The main difficulty to apply Theorem 2.15 is to show that a weak solution of (2.17) verifies some boundary condition. In the following result, we give a sufficient condition.

Proposition 2.17 (**Regularity**). *Let Ω be a nonempty bounded open subset of $\mathbb{R}^N$ with a $C^{1,1}$ boundary, let ν be the outward unit normal vector to Γ, let $V \in L^N(\Omega;\mathbb{C})$ ($V \in L^{2+\varepsilon}(\Omega;\mathbb{C})$, for some $\varepsilon > 0$, if $N = 2$ and $V \in L^2(\Omega;\mathbb{C})$ if $N = 1$), let $0 < m < 1$, let $a \in \mathbb{C}$ and let $F \in L^2(\Omega)$.*

1) *Let $u \in H^1_0(\Omega)$ be any weak solution to (2.17) and (1.3). Then $u \in H^2(\Omega)$ and $u_{|\Gamma} = 0$ in the sense of the trace.*
2) *Let $u \in H^1(\Omega)$ be any weak solution to (2.17) and (1.4). Then $u \in H^2(\Omega)$ and $\frac{\partial u}{\partial \nu}_{|\Gamma} = 0$ in the sense of the trace.*

3 Some useful estimates

Lemma 3.1 (**Poincaré's inequality**). *Let Ω a nonempty subset of $\mathbb{R}^N$ be such that $|\Omega| < \infty$ and let $1 \leqslant p < \infty$. Then,*

$$\forall u \in W^{1,p}_0(\Omega),\ \|u\|_{L^p(\Omega)} \leqslant C_{\mathrm{P}}(p)\|\nabla u\|_{L^p(\Omega)}, \tag{3.1}$$

where $C_{\mathrm{P}}(p) = C_{\mathrm{P}}(|\Omega|, N, p)$. In particular,

$$\forall u \in H^1_0(\Omega),\ \|u\|_{L^2(\Omega)} \leqslant C_{\mathrm{P}}\|\nabla u\|_{L^2(\Omega)}, \tag{3.2}$$

where $C_{\mathrm{P}} = C_{\mathrm{P}}(2) = C_{\mathrm{P}}(|\Omega|, N)$.

This result is well-known but for convenience, we briefly recall its proof.

Proof of Lemma 3.1. By density, it is sufficient to establish (3.1) for $\varphi \in \mathscr{D}(\Omega)$. Applying Hölder's inequality, we obtain

$$\|\varphi\|_{L^p(\Omega)} \leqslant |\Omega|^{\frac{1}{N+p}}\|\varphi\|_{L^{\frac{p(N+p)}{N}}(\Omega)}. \tag{3.3}$$

Extending φ by 0 outside of Ω and applying Gagliardo-Nirenberg's inequality (see for instance Cazenave [7], Theorem 5.4.9, p.155), we get

$$\|\varphi\|_{L^{\frac{p(N+p)}{N}}(\Omega)} \leqslant C(N,p)\|\nabla\varphi\|^{\frac{N}{N+p}}_{L^p(\Omega)}\|\varphi\|^{\frac{p}{N+p}}_{L^p(\Omega)}. \tag{3.4}$$

Estimate (3.1) then follows from (3.3) and (3.4), and (3.2) comes from (3.1) applied with $p = 2$. □

We will frequently use the following estimate which comes from Hölder's inequality. Let Ω a nonempty subset of $\mathbb{R}^N$ be such that $|\Omega| < \infty$ and let $0 \leqslant m \leqslant 1$. Then, $L^2(\Omega) \hookrightarrow L^{m+1}(\Omega)$ and

$$\forall u \in L^{m+1}(\Omega),\ \|u\|^{m+1}_{L^{m+1}(\Omega)} \leqslant |\Omega|^{\frac{1-m}{2}}\|u\|^{m+1}_{L^2(\Omega)}. \tag{3.5}$$

We recall the well-known Young's inequality. For any real $x \geqslant 0$, $y \geqslant 0$ and $\mu > 0$, one has

$$xy \leqslant \frac{\mu^2}{2}x^2 + \frac{1}{2\mu^2}y^2. \tag{3.6}$$

In particular,

$$xy \leqslant \frac{C_{\mathrm{P}}^2}{2}x^2 + \frac{C_{\mathrm{P}}^{-2}}{2}y^2, \tag{3.7}$$

where C_{P} is the constant in (3.2).

4 A priori estimates

Lemma 4.1. *Let Ω a nonempty open subset of $\mathbb{R}^N$ be such that $|\Omega| < \infty$, let ω a nonempty open subset of $\mathbb{R}^N$ be such that $\omega \subseteq \Omega$, let $0 \leqslant m \leqslant 1$, let $(a,b) \in \mathbb{C}^2$, let $\alpha, \beta \geqslant 0$ and let $F \in L^2(\Omega)$. Let $u \in H_0^1(\Omega)$ satisfies*

$$\Big| \|\nabla u\|_{L^2(\Omega)}^2 + \mathrm{Re}(a)\left(\|u\|_{L^{m+1}(\omega)}^{m+1} + \alpha\|u\|_{L^1(\omega^{\mathrm{c}})}\right) \\ + \mathrm{Re}(b)\left(\|u\|_{L^2(\omega)}^2 + \beta\|u\|_{L^1(\omega^{\mathrm{c}})}\right)\Big| \leqslant \int_\Omega |Fu|\mathrm{d}x, \tag{4.1}$$

$$\left|\mathrm{Im}(a)\left(\|u\|_{L^{m+1}(\omega)}^{m+1} + \alpha\|u\|_{L^1(\omega^{\mathrm{c}})}\right) + \mathrm{Im}(b)\left(\|u\|_{L^2(\omega)}^2 + \beta\|u\|_{L^1(\omega^{\mathrm{c}})}\right)\right| \leqslant \int_\Omega |Fu|\mathrm{d}x. \tag{4.2}$$

Assume that one of the four following assertions holds.

1) $\mathrm{Re}(b) > 0$ *and one of the two following assertions holds.*
   a) $|\omega| = |\Omega|$.
   b) $|\omega| < |\Omega|$, $F \in L^\infty(\Omega)$ *and* $\alpha\mathrm{Re}(a) + \beta\mathrm{Re}(b) > \|F\|_{L^\infty(\Omega)}$.
2) $\mathrm{Re}(b) = 0$. *If* $\mathrm{Re}(a) < 0$ *then assume further that* $\alpha\|u\|_{L^1(\omega^{\mathrm{c}})} \leqslant \|u\|_{L^{m+1}(\omega^{\mathrm{c}})}^{m+1}$.
3) $\mathrm{Re}(b) < 0$, $\mathrm{Im}(b) \neq 0$ *and one of the two following assertions holds.*
   a) $|\omega| = |\Omega|$.
   b) $|\omega| < |\Omega|$, $F \in L^\infty(\Omega)$, $-\alpha|\mathrm{Im}(a)| + \frac{\beta}{2}|\mathrm{Im}(b)| > \|F\|_{L^\infty(\Omega)}$ *and* $\alpha\|u\|_{L^1(\omega^{\mathrm{c}})} \leqslant \|u\|_{L^{m+1}(\omega^{\mathrm{c}})}^{m+1}$.
4) $-C_{\mathrm{P}}^{-2} < \mathrm{Re}(b) < 0$, *where C_{P} is the constant in* (3.2), $\alpha\|u\|_{L^1(\omega^{\mathrm{c}})} \leqslant \|u\|_{L^{m+1}(\omega^{\mathrm{c}})}^{m+1}$ *and* $\beta\|u\|_{L^1(\omega^{\mathrm{c}})} \leqslant \|u\|_{L^2(\omega^{\mathrm{c}})}^2$.

Then we have the following estimate.

$$\|u\|_{H_0^1(\Omega)} \leqslant C, \tag{4.3}$$

where $C = C(\|F\|_{L^2(\Omega)}, |\Omega|, |a|, |b|, N, m)$.

Remark 4.2. Note that when $|\omega| = |\Omega|$, assumptions as $\alpha\|u\|_{L^1(\omega^c)} \leqslant \|u\|^{m+1}_{L^{m+1}(\omega^c)}$ and $\beta\|u\|_{L^1(\omega^c)} \leqslant \|u\|^2_{L^2(\omega^c)}$ are automatically fulfilled, for any $\alpha, \beta \geqslant 0$.

Proof of Lemma 4.1. By Poincaré's inequality (3.2), it is sufficient to establish

$$\|\nabla u\|_{L^2(\Omega)} \leqslant C(\|F\|_{L^2(\Omega)}, |\Omega|, |a|, |b|, N, m). \tag{4.4}$$

We divide the proof in 4 steps.

Step 1. Proof of (4.4) with Assumption 1).

Let $A = \alpha\mathrm{Re}(a) + \beta\mathrm{Re}(b)$. Applying Hölder's inequality (3.5) in (4.1), we get

$$\begin{aligned}
&\|\nabla u\|^2_{L^2(\Omega)} + \left(\mathrm{Re}(b)\|u\|^{1-m}_{L^2(\omega)} - |\mathrm{Re}(a)||\Omega|^{\frac{1-m}{2}}\right)\|u\|^{m+1}_{L^2(\omega)} + A\|u\|_{L^1(\omega^c)}\\
&\leqslant \|\nabla u\|^2_{L^2(\Omega)} + \mathrm{Re}(a)\left(\|u\|^{m+1}_{L^{m+1}(\omega)} + \alpha\|u\|_{L^1(\omega^c)}\right) + \mathrm{Re}(b)\left(\|u\|^2_{L^2(\omega)} + \beta\|u\|_{L^1(\omega^c)}\right)\\
&\leqslant \int_\Omega |Fu|\mathrm{d}x.
\end{aligned} \tag{4.5}$$

Case 1. $\mathrm{Re}(b)\|u\|^{1-m}_{L^2(\omega)} - |\mathrm{Re}(a)||\Omega|^{\frac{1-m}{2}} \leqslant 1.$

It then follows that $\|u\|_{L^2(\omega)} \leqslant C(|\Omega|, |a|, |b|, m)$. By (4.5),

$$\|\nabla u\|^2_{L^2(\Omega)} + A\|u\|_{L^1(\omega^c)} \leqslant |\mathrm{Re}(a)||\Omega|^{\frac{1-m}{2}}\|u\|^{m+1}_{L^2(\omega)} + \int_\Omega |Fu|\mathrm{d}x. \tag{4.6}$$

When $|\omega| = |\Omega|$, we have $A\|u\|_{L^1(\omega^c)} = 0$ and $\|u\|_{L^2(\Omega)} \leqslant C(|\Omega|, |a|, |b|, m)$. Thus, estimate (4.6), with help of Cauchy-Schwarz's inequality, becomes

$$\|\nabla u\|^2_{L^2(\Omega)} \leqslant |\mathrm{Re}(a)||\Omega|^{\frac{1-m}{2}}\|u\|^{m+1}_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leqslant C(\|F\|_{L^2(\Omega)}, |\Omega|, |a|, |b|, m),$$

which is (4.4). Now, assume that $|\omega| < |\Omega|$. We apply Hölder's inequality in (4.5) to obtain,

$$\begin{aligned}
&\|\nabla u\|^2_{L^2(\Omega)} + A\|u\|_{L^1(\omega^c)} \leqslant |\mathrm{Re}(a)||\Omega|^{\frac{1-m}{2}}\|u\|^{m+1}_{L^2(\omega)} + \int_\Omega |Fu|\mathrm{d}x\\
&\leqslant |\mathrm{Re}(a)||\Omega|^{\frac{1-m}{2}}\|u\|^{m+1}_{L^2(\omega)} + \|F\|_{L^2(\omega)}\|u\|_{L^2(\omega)} + \|F\|_{L^\infty(\omega^c)}\|u\|_{L^1(\omega^c)},\\
&\leqslant C(\|F\|_{L^2(\Omega)}, |\Omega|, |a|, |b|, m) + A\|u\|_{L^1(\omega^c)},
\end{aligned}$$

from which we deduce (4.4).

Case 2. $\mathrm{Re}(b)\|u\|^{1-m}_{L^2(\omega)} - |\mathrm{Re}(a)||\Omega|^{\frac{1-m}{2}} > 1.$

Applying Young's inequality (3.7), Hölder's inequality and Poincaré's inequality (3.2) in (4.5), we get

$$\begin{aligned}
&\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^{m+1}_{L^2(\Omega)}\\
&\leqslant \frac{C_\mathrm{P}^2}{2}\|F\|^2_{L^2(\Omega)} + \frac{C_\mathrm{P}^{-2}}{2}\|u\|^2_{L^2(\Omega)}\\
&\leqslant \frac{C_\mathrm{P}^2}{2}\|F\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)},
\end{aligned}$$

if $|\omega|=|\Omega|$ and

$$\begin{aligned}
&\|\nabla u\|_{L^2(\Omega)}^2+\|u\|_{L^2(\omega)}^{m+1}+A\|u\|_{L^1(\omega^c)}\\
\leqslant&\ \frac{C_{\mathrm{P}}^2}{2}\|F\|_{L^2(\omega)}^2+\frac{C_{\mathrm{P}}^{-2}}{2}\|u\|_{L^2(\omega)}^2+\|F\|_{L^\infty(\omega^c)}\|u\|_{L^1(\omega^c)}\\
\leqslant&\ \frac{C_{\mathrm{P}}^2}{2}\|F\|_{L^2(\Omega)}^2+\frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2+A\|u\|_{L^1(\omega^c)},
\end{aligned}$$

if $|\omega|<|\Omega|$. We get in both cases, $\|\nabla u\|_{L^2(\Omega)}\leqslant C_{\mathrm{P}}\|F\|_{L^2(\Omega)}$. Then (4.4) holds and Step 1 is proved.

Step 2. Proof of (4.4) with Assumption 2).

Case 1. $\mathrm{Re}(a)\geqslant 0$.

We apply Young's inequality (3.7) and Poincaré's inequality (3.2) in (4.1). It follows that, $\|\nabla u\|_{L^2(\Omega)}^2\leqslant \frac{C_{\mathrm{P}}^2}{2}\|F\|_{L^2(\Omega)}^2+\frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2$. We then have (4.4).

Case 2. $\mathrm{Re}(a)<0$.

By Assumption 2) and (4.1), we have

$$\|\nabla u\|_{L^2(\Omega)}^2\leqslant|\mathrm{Re}(a)|\|u\|_{L^{m+1}(\Omega)}^{m+1}+\int_\Omega|Fu|\mathrm{d}x.$$

Using Hölder's inequality (3.5), Young's inequality (3.7) and Poincaré's inequality (3.2), we get

$$\begin{aligned}
&\|\nabla u\|_{L^2(\Omega)}^2\\
\leqslant&\ |\mathrm{Re}(a)||\Omega|^{\frac{1-m}{2}}\|u\|_{L^2(\Omega)}^{m+1}+\frac{C_{\mathrm{P}}^2}{2}\|F\|_{L^2(\Omega)}^2+\frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2\\
\leqslant&\ |\mathrm{Re}(a)||\Omega|^{\frac{1-m}{2}}C_{\mathrm{P}}^{m+1}\|\nabla u\|_{L^2(\Omega)}^{m+1}+\frac{C_{\mathrm{P}}^2}{2}\|F\|_{L^2(\Omega)}^2+\frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2,
\end{aligned}$$

which yields,

$$\left(\|\nabla u\|_{L^2(\Omega)}^{1-m}-C\right)\|\nabla u\|_{L^2(\Omega)}^{m+1}\leqslant C_{\mathrm{P}}^2\|F\|_{L^2(\Omega)}^2,$$

where $C=C(|\Omega|,|a|,N,m)$. Both cases,

$$\begin{cases}
\|\nabla u\|_{L^2(\Omega)}^{1-m}-C\leqslant 1,\\
\text{or}\\
\|\nabla u\|_{L^2(\Omega)}^{1-m}-C>1,
\end{cases}$$

lead to (4.4).

Step 3. Proof of (4.4) with Assumption 3).

From (4.2), (3.5), Young's inequality (3.6) with $x=|F|$, $y=|u|$, $\mu=|\mathrm{Im}(b)|^{-\frac{1}{2}}$ and Hölder's

inequality, we get

$$\begin{aligned}&|\mathrm{Im}(b)|\left(\|u\|_{L^2(\omega)}^2+\beta\|u\|_{L^1(\omega^c)}\right)\\&\leqslant|\mathrm{Im}(a)||\Omega|^{\frac{1-m}{2}}\|u\|_{L^2(\omega)}^{m+1}+\alpha|\mathrm{Im}(a)|\|u\|_{L^1(\omega^c)}\\&+\frac{1}{2|\mathrm{Im}(b)|}\|F\|_{L^2(\omega)}^2+\frac{|\mathrm{Im}(b)|}{2}\|u\|_{L^2(\omega)}^2+\|F\|_{L^\infty(\omega^c)}\|u\|_{L^1(\omega^c)},\end{aligned}$$

which yields,

$$\begin{aligned}&\left(|\mathrm{Im}(b)|\|u\|_{L^2(\omega)}^{1-m}-2|\mathrm{Im}(a)||\Omega|^{\frac{1-m}{2}}\right)\|u\|_{L^2(\omega)}^{m+1}+2\big(\beta|\mathrm{Im}(b)|-\alpha|\mathrm{Im}(a)|-\|F\|_{L^\infty(\omega^c)}\big)\|u\|_{L^1(\omega^c)}\\&\leqslant\frac{1}{|\mathrm{Im}(b)|}\|F\|_{L^2(\omega)}^2.\end{aligned}$$

Obtaining easily (4.4) when $|\omega|=|\Omega|$ with help of (4.1) (by following the method of Step 1), we may assume that $|\omega|<|\Omega|$. It then follows from Assumption 3)b) that,

$$\left(|\mathrm{Im}(b)|\|u\|_{L^2(\omega)}^{1-m}-2|\mathrm{Im}(a)||\Omega|^{\frac{1-m}{2}}\right)\|u\|_{L^2(\omega)}^{m+1}+\beta|\mathrm{Im}(b)|\|u\|_{L^1(\omega^c)}\leqslant\frac{1}{|\mathrm{Im}(b)|}\|F\|_{L^2(\omega)}^2. \tag{4.7}$$

Case 1. $|\mathrm{Im}(b)|\|u\|_{L^2(\omega)}^{1-m}-2|\mathrm{Im}(a)||\Omega|^{\frac{1-m}{2}}\leqslant 1.$

It follows that

$$\|u\|_{L^2(\omega)}\leqslant C(\|F\|_{L^2(\Omega)},|\Omega|,|a|,|b|,m), \tag{4.8}$$

and by (4.2), (3.5), Hölder's inequality and Assumption 3)b), one obtains

$$\begin{aligned}&\big(\beta|\mathrm{Im}(b)|-\alpha|\mathrm{Im}(a)|\big)\|u\|_{L^1(\omega^c)}\\&\leqslant|\mathrm{Im}(a)||\Omega|^{\frac{1-m}{2}}\|u\|_{L^2(\omega)}^{m+1}+|\mathrm{Im}(b)|\|u\|_{L^2(\omega)}^2+\|F\|_{L^2(\omega)}\|u\|_{L^2(\omega)}+\|F\|_{L^\infty(\omega^c)}\|u\|_{L^1(\omega^c)},\\&\leqslant C(\|F\|_{L^2(\Omega)},|\Omega|,|a|,|b|,m)+\left(\frac{\beta}{2}|\mathrm{Im}(b)|-\alpha|\mathrm{Im}(a)|\right)\|u\|_{L^1(\omega^c)},\end{aligned}$$

so that,

$$\beta\|u\|_{L^1(\omega^c)}\leqslant C(\|F\|_{L^2(\Omega)},|\Omega|,|a|,|b|,m). \tag{4.9}$$

Case 2. $|\mathrm{Im}(b)|\|u\|_{L^2(\omega)}^{1-m}-2|\mathrm{Im}(a)||\Omega|^{\frac{1-m}{2}}>1.$

Then estimates (4.7) implies that

$$\|u\|_{L^2(\omega)}^{m+1}+\beta|\mathrm{Im}(b)|\|u\|_{L^1(\omega^c)}\leqslant\frac{1}{|\mathrm{Im}(b)|}\|F\|_{L^2(\omega)}^2,$$

which yields to (4.8) and (4.9). So in both cases, estimates (4.8) and (4.9) hold.

Finally, by (4.1), Assumption 3)b), Hölder's inequality (3.5), Young's inequality (3.7), Poincaré's

inequality (3.2), (4.8) and (4.9), one obtains

$$
\begin{aligned}
&\|\nabla u\|^2_{L^2(\Omega)} \\
&\leqslant |\mathrm{Re}(a)||\Omega|^{\frac{1-m}{2}}\|u\|^{m+1}_{L^2(\Omega)} + |\mathrm{Re}(b)|\big(\|u\|^2_{L^2(\omega)} + \beta\|u\|_{L^1(\omega^c)}\big) + \frac{C_{\mathrm{P}}^2}{2}\|F\|^2_{L^2(\Omega)} + \frac{C_{\mathrm{P}}^{-2}}{2}\|u\|^2_{L^2(\Omega)} \\
&\leqslant C_0\|\nabla u\|^{m+1}_{L^2(\Omega)} + C_1 + \frac{C_{\mathrm{P}}^2}{2}\|F\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)},
\end{aligned}
$$

where $C_0 = C_0(|\Omega|, |a|, N, m)$ and $C_1 = C_1(\|F\|_{L^2(\Omega)}, |\Omega|, |a|, |b|, m)$. It follows that,

$$
\big(\|\nabla u\|^{1-m}_{L^2(\Omega)} - 2C_0\big)\|\nabla u\|^{m+1}_{L^2(\Omega)} \leqslant 2C_1 + C_{\mathrm{P}}^2\|F\|^2_{L^2(\Omega)} \leqslant C(\|F\|_{L^2(\Omega)}, |\Omega|, |a|, |b|, N, m).
$$

We then easily deduce (4.4) and Step 3 is proved.

Step 4. Proof of (4.4) with Assumption 4).

We use Assumption 4), Hölder's inequality (3.5), Young's inequality (3.6) with $x = |F|$, $y = |u|$, and Poincaré's inequality (3.2) in (4.1) to get,

$$
\begin{aligned}
&\|\nabla u\|^2_{L^2(\Omega)} \\
&\leqslant |\mathrm{Re}(a)||\Omega|^{\frac{1-m}{2}}\|u\|^{m+1}_{L^2(\Omega)} + |\mathrm{Re}(b)|\|u\|^2_{L^2(\Omega)} + \frac{\mu^2}{2}\|F\|^2_{L^2(\Omega)} + \frac{1}{2\mu^2}\|u\|^2_{L^2(\Omega)} \\
&\leqslant C\|\nabla u\|^{m+1}_{L^2(\Omega)} + \left(|\mathrm{Re}(b)|C_{\mathrm{P}}^2 + \frac{C_{\mathrm{P}}^2}{2\mu^2}\right)\|\nabla u\|^2_{L^2(\Omega)} + \frac{\mu^2}{2}\|F\|^2_{L^2(\Omega)}.
\end{aligned}
$$

where $C = C(|\Omega|, |a|, N, m)$. We then deduce,

$$
\left(\left(1 - |\mathrm{Re}(b)|C_{\mathrm{P}}^2 - \frac{C_{\mathrm{P}}^2}{2\mu^2}\right)\|\nabla u\|^{1-m}_{L^2(\Omega)} - C\right)\|\nabla u\|^{m+1}_{L^2(\Omega)} \leqslant \frac{\mu^2}{2}\|F\|^2_{L^2(\Omega)}.
$$

Since $|\mathrm{Re}(b)| < C_{\mathrm{P}}^{-2}$, we have $1 - |\mathrm{Re}(b)|C_{\mathrm{P}}^2 > 0$ and so there exists $\mu_0 = \mu_0(|\Omega|, |b|, N) > 0$ large enough such that

$$
1 - |\mathrm{Re}(b)|C_{\mathrm{P}}^2 - \frac{C_{\mathrm{P}}^2}{2\mu_0^2} > 0.
$$

For such a μ_0, it follows that,

$$
\left(C_0\|\nabla u\|^{1-m}_{L^2(\Omega)} - C\right)\|\nabla u\|^{m+1}_{L^2(\Omega)} \leqslant \frac{\mu_0^2}{2}\|F\|^2_{L^2(\Omega)},
$$

where $C_0 = 1 - |\mathrm{Re}(b)|C_{\mathrm{P}}^2 - \frac{C_{\mathrm{P}}^2}{2\mu_0^2}$. Note that $C_0 = C_0(|\Omega|, |b|, N)$. We then easily deduce (4.4) and Step 4 is proved. This concludes the proof of the lemma. □

Corollary 4.3. *Let $(\Omega_n)_{n\in\mathbb{N}}$ a sequence of nonempty open subsets of $\mathbb{R}^N$ be such that $\sup\limits_{n\in\mathbb{N}}|\Omega_n| < \infty$, let $0 < m < 1$, let $(a,b) \in \mathbb{C}^2$ and let $(F_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega_n)$ be such that $\sup\limits_{n\in\mathbb{N}}\|F_n\|_{L^2(\Omega_n)} < \infty$. If*

$\mathrm{Re}(b) < 0$ *then assume further that* $\mathrm{Im}(b) \neq 0$ *or* $-\frac{1}{C_{\mathrm{P}}^2} < \mathrm{Re}(b)$, *where* C_{P} *is the constant in* (3.2) *of Lemma* 3.1. *Let* $(u_\ell^n)_{(n,\ell)\in\mathbb{N}^2} \subset H_0^1(\Omega_n)$ *be a sequence satisfying*

$$\forall n \in \mathbb{N},\ \forall \ell \in \mathbb{N},\ -\Delta u_\ell^n + f_\ell\big(u_\ell^n\big) = F_n,\ \text{in } L^2(\Omega_n), \tag{4.10}$$

where for any $\ell \in \mathbb{N}$,

$$\forall u \in L^2(\Omega_n),\ f_\ell(u) = \begin{cases} a|u|^{-(1-m)}u + bu, & \text{if } |u| \leqslant \ell, \\ a\ell^m \dfrac{u}{|u|} + b\ell \dfrac{u}{|u|}, & \text{if } |u| > \ell. \end{cases} \tag{4.11}$$

Then there exists a diagonal extraction $\left(u_{\varphi(n)}^n\right)_{n\in\mathbb{N}}$ *of* $(u_\ell^n)_{(n,\ell)\in\mathbb{N}^2}$ *such that the following estimate holds.*

$$\forall n \in \mathbb{N},\ \big\|u_{\varphi(n)}^n\big\|_{H_0^1(\Omega_n)} \leqslant C,$$

where $C = C\left(\sup\limits_{n\in\mathbb{N}} \|F_n\|_{L^2(\Omega_n)}, \sup\limits_{n\in\mathbb{N}} |\Omega_n|, |a|, |b|, N, m\right)$.

Proof. One easily sees that $(f_\ell)_{\ell\in\mathbb{N}} \subset C\big(L^2(\Omega); L^2(\Omega)\big)$ and it follows that u_ℓ^n and $\mathrm{i}u_\ell^n$ are admissible test functions in (2.2). We then get,

$$\begin{aligned} \|\nabla u_\ell^n\|_{L^2(\Omega_n)}^2 + \mathrm{Re}(a)\left(\|u_\ell^n\|_{L^{m+1}(\{|u_\ell^n|\leqslant\ell\})}^{m+1} + \ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) \\ + \mathrm{Re}(b)\left(\|u_\ell^n\|_{L^2(\{|u_\ell^n|\leqslant\ell\})}^2 + \ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) = \mathrm{Re}\int_{\Omega_n} F_n\overline{u_\ell^n}\mathrm{d}x, \end{aligned}$$

$$\begin{aligned} \mathrm{Im}(a)\left(\|u_\ell^n\|_{L^{m+1}(\{|u_\ell^n|\leqslant\ell\})}^{m+1} + \ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) \\ + \mathrm{Im}(b)\left(\|u_\ell^n\|_{L^2(\{|u_\ell^n|\leqslant\ell\})}^2 + \ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) = \mathrm{Im}\int_{\Omega_n} F_n\overline{u_\ell^n}\mathrm{d}x, \end{aligned}$$

for any $(n,\ell) \in \mathbb{N}^2$. We first note that,

$$\forall (n,\ell) \in \mathbb{N}^2, \begin{cases} \ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})} \leqslant \|u_\ell^n\|_{L^{m+1}(\{|u_\ell^n|>\ell\})}^{m+1}, \\ \ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})} \leqslant \|u_\ell^n\|_{L^2(\{|u_\ell^n|>\ell\})}^2, \end{cases} \tag{4.12}$$

For each $n \in \mathbb{N}$, let $\varphi(n) \in \mathbb{N}$ be large enough to have

$$\varphi(n)^{1-m} > \begin{cases} \dfrac{\|F_n\|_{L^\infty(\Omega_n)} + |\mathrm{Re}(a)|}{\mathrm{Re}(b)}, & \text{if } \mathrm{Re}(b) > 0, \\ 2\dfrac{\|F_n\|_{L^\infty(\Omega_n)} + |\mathrm{Im}(a)|}{|\mathrm{Im}(b)|}, & \text{if } \mathrm{Im}(b) \neq 0. \end{cases}$$

It follows that for any $n \in \mathbb{N}$,

$$\|F_n\|_{L^\infty(\Omega_n)} < \begin{cases} \varphi(n)^m\mathrm{Re}(a) + \varphi(n)\mathrm{Re}(b), & \text{if } \mathrm{Re}(b) > 0, \\ -\varphi(n)^m|\mathrm{Im}(a)| + \dfrac{\varphi(n)}{2}|\mathrm{Im}(b)|, & \text{if } \mathrm{Im}(b) \neq 0. \end{cases} \tag{4.13}$$

If $\mathrm{Im}(b)=0$ and $\mathrm{Re}(b)\leqslant 0$ then we choose $\varphi(n)=n$. For each $n\in\mathbb{N}$, with help of (4.12) and (4.13), we may apply Lemma 4.1 to $u^n_{\varphi(n)}$ with $\omega=\left\{x\in\Omega_n;\left|u^n_{\varphi(n)}(x)\right|\leqslant\varphi(n)\right\}$, $\alpha=\varphi(n)^m$ and $\beta=\varphi(n)$. Hence the result. $\square$

Lemma 4.4. *Let $\Omega\subseteq\mathbb{R}^N$ be a nonempty open subset, let ω a nonempty open subset of $\mathbb{R}^N$ be such that $\omega\subseteq\Omega$, let $m\geqslant 0$ and let $(a,b,c)\in\mathbb{C}^3$ be such that $\mathrm{Im}(b)\neq 0$. If $\mathrm{Re}(a)\leqslant 0$ then assume further that $\mathrm{Im}(a)\neq 0$. Let $\alpha,\beta,R\geqslant 0$, let $F\in L^2(\Omega)$ and let*

$$A=\begin{cases}\max\left\{1,\frac{1+|b|+R^2|c|}{|\mathrm{Im}(b)|},\frac{|\mathrm{Re}(a)|}{|\mathrm{Im}(a)|}\right\}, & \text{if } \mathrm{Re}(a)\leqslant 0,\\ \max\left\{1,\frac{1+|b|+R^2|c|}{|\mathrm{Im}(b)|}\right\}, & \text{if } \mathrm{Re}(a)>0.\end{cases}$$

If $|\omega|<|\Omega|$ then assume further that $F\in L^\infty(\Omega)$ and $\beta\geqslant 2A\|F\|_{L^\infty(\Omega)}+1$. Let $u\in H^1(\Omega)$ satisfies

$$\|\nabla u\|^2_{L^2(\Omega)}+\mathrm{Re}(a)\left(\|u\|^{m+1}_{L^{m+1}(\omega)}+\alpha\|u\|_{L^1(\omega^c)}\right) - (|b|+R^2|c|)\left(\|u\|^2_{L^2(\omega)}+\beta\|u\|_{L^1(\omega^c)}\right)\leqslant\int_\Omega|Fu|\mathrm{d}x, \tag{4.14}$$

$$|\mathrm{Im}(a)|\left(\|u\|^{m+1}_{L^{m+1}(\omega)}+\alpha\|u\|_{L^1(\omega^c)}\right)+|\mathrm{Im}(b)|\left(\|u\|^2_{L^2(\omega)}+\beta\|u\|_{L^1(\omega^c)}\right)\leqslant\int_\Omega|Fu|\mathrm{d}x. \tag{4.15}$$

Then there exists a positive constant $M=M(|a|,|b|,|c|)$ such that,

$$\|\nabla u\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\omega)}+\|u\|^{m+1}_{L^{m+1}(\omega)}+\|u\|_{L^1(\omega^c)}\leqslant M(R^4+1)\|F\|^2_{L^2(\Omega)}. \tag{4.16}$$

Proof of Lemma 4.4. Let A be as in the lemma. We multiply (4.15) by A and sum the result to (4.14). This yields,

$$\|\nabla u\|^2_{L^2(\Omega)}+A_0\left(\|u\|^{m+1}_{L^{m+1}(\omega)}+\alpha\|u\|_{L^1(\omega^c)}\right)+\|u\|^2_{L^2(\omega)}+\beta\|u\|_{L^1(\omega^c)}\leqslant 2A\int_\Omega|Fu|\mathrm{d}x,$$

where $A_0=A|\mathrm{Im}(a)|+\mathrm{Re}(a)$. Applying Hölder's inequality and Young's inequality (3.6) with $x=|F|$, $y=|u|$ and $\mu=\sqrt{2A}$, we get

$$\|\nabla u\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\omega)}+A_0\|u\|^{m+1}_{L^{m+1}(\omega)}+\beta\|u\|_{L^1(\omega^c)} \leqslant 2A\|F\|_{L^\infty(\Omega)}\|u\|_{L^1(\omega^c)}+2A^2\|F\|^2_{L^2(\Omega)}+\frac{1}{2}\|u\|^2_{L^2(\omega)},$$

from which we deduce the result if $|\omega|=|\Omega|$. Now, suppose $|\omega|<|\Omega|$. The above estimate then leads to,

$$\|\nabla u\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\omega)}+A_0\|u\|^{m+1}_{L^{m+1}(\omega)}+\left(\beta-2A\|F\|_{L^\infty(\Omega)}\right)\|u\|_{L^1(\omega^c)}\leqslant 4A^2\|F\|^2_{L^2(\Omega)},$$

from which we prove the lemma since $\beta-2A\|F\|_{L^\infty(\Omega)}\geqslant 1$. $\square$

Lemma 4.5. *Let $(a,b) \in \mathbb{A}^2$ satisfies (2.15). Then there exists $\delta_\star = \delta_\star(|a|,|b|) \in (0,1]$, $L = L(|a|,|b|)$ and $M = M(|a|,|b|)$ satisfying the following property. If $\delta \in [0,\delta_\star]$, C_0, C_1, C_2, C_3, C_4 are six nonnegative real numbers satisfying*

$$\left|C_1 + \delta C_2 + \mathrm{Re}(a)C_3 + \big(\mathrm{Re}(b) - \delta\big)C_4\right| \leqslant C_0, \tag{4.17}$$

$$\left|\mathrm{Im}(a)C_3 + \mathrm{Im}(b)C_4\right| \leqslant C_0, \tag{4.18}$$

then

$$0 \leqslant C_1 + LC_3 + LC_4 \leqslant MC_0. \tag{4.19}$$

Proof. We split the proof in 4 cases. Let $\gamma > 0$ be small enough to be chosen later. Note that when $\mathrm{Im}(a)\mathrm{Im}(b) \geqslant 0$ then estimate (4.18) can be rewritten as

$$|\mathrm{Im}(a)|C_3 + |\mathrm{Im}(b)|C_4 \leqslant C_0. \tag{4.20}$$

Case 1. $\mathrm{Re}(a) \geqslant 0$, $\mathrm{Re}(b) \geqslant 0$ and $\mathrm{Im}(a)\mathrm{Im}(b) \geqslant 0$.

We add (4.20) with (4.17) and obtain,

$$C_1 + \big(\mathrm{Re}(a) + |\mathrm{Im}(a)|\big)C_3 + \big(\mathrm{Re}(b) - \delta_\star + |\mathrm{Im}(b)|\big)C_4 \leqslant 2C_0.$$

Case 2. $\Big(\mathrm{Re}(a) \geqslant 0$, $\mathrm{Re}(b) < 0$ and $\mathrm{Im}(a)\mathrm{Im}(b) \geqslant 0\Big)$ or $\Big(\mathrm{Im}(a)\mathrm{Im}(b) < 0\Big)$.

We compute (4.17) $- \frac{\mathrm{Re}(b)-\gamma}{\mathrm{Im}(b)}$(4.18) to obtain

$$C_1 + \frac{\mathrm{Re}(a)\mathrm{Im}(b) - \mathrm{Re}(b)\mathrm{Im}(a) + \gamma\mathrm{Im}(a)}{\mathrm{Im}(b)}C_3 + (\gamma - \delta_\star)C_4 \leqslant \frac{|\mathrm{Re}(b)| + |\mathrm{Im}(b)| + \gamma}{|\mathrm{Im}(b)|}C_0.$$

Case 3. $\mathrm{Re}(a) < 0$, $\mathrm{Re}(b) \geqslant 0$ and $\mathrm{Im}(a)\mathrm{Im}(b) \geqslant 0$.

We compute (4.17) $- \frac{\mathrm{Re}(a)-\gamma}{\mathrm{Im}(a)}$(4.18) to obtain,

$$C_1 + \gamma C_3 + \left(\frac{\mathrm{Re}(b)\mathrm{Im}(a) - \mathrm{Re}(a)\mathrm{Im}(b) + \gamma\mathrm{Im}(b)}{\mathrm{Im}(a)} - \delta_\star\right)C_4 \leqslant \frac{|\mathrm{Re}(a)| + |\mathrm{Im}(a)| + \gamma}{|\mathrm{Im}(a)|}C_0.$$

Case 4. $\mathrm{Re}(a) < 0$, $\mathrm{Re}(b) < 0$ and $\mathrm{Im}(a)\mathrm{Im}(b) \geqslant 0$.

Note that by assumptions $(a,b) \in \mathbb{A}^2$, $\mathrm{Re}(a) < 0$ and $\mathrm{Re}(b) < 0$, one necessarily has $\mathrm{Im}(a) \neq 0$ and $\mathrm{Im}(b) \neq 0$. Thus, we can compute (4.17) $+ \max\left\{\frac{|\mathrm{Re}(a)|+\gamma}{|\mathrm{Im}(a)|}, \frac{|\mathrm{Re}(b)|+\gamma}{|\mathrm{Im}(b)|}\right\}$(4.20) to obtain,

$$C_1 + \gamma C_3 + (\gamma - \delta_\star)C_4 \leqslant \left(\frac{|\mathrm{Re}(a)| + |\mathrm{Im}(a)| + \gamma}{|\mathrm{Im}(a)|} + \frac{|\mathrm{Re}(b)| + |\mathrm{Im}(b)| + \gamma}{|\mathrm{Im}(b)|}\right)C_0.$$

In both cases, we may choose $\gamma > 0$ small enough to have

$$\begin{cases} \dfrac{\mathrm{Re}(a)\mathrm{Im}(b) - \mathrm{Re}(b)\mathrm{Im}(a) + \gamma\mathrm{Im}(a)}{\mathrm{Im}(b)} > 0, & \text{in Case 2,} \\[2ex] \dfrac{\mathrm{Re}(b)\mathrm{Im}(a) - \mathrm{Re}(a)\mathrm{Im}(b) + \gamma\mathrm{Im}(b)}{\mathrm{Im}(a)} > 0, & \text{in Case 3.} \end{cases}$$

Then we fix $0 < \delta_\star < \min\{1, \gamma, |\mathrm{Im}(b)| + |\mathrm{Re}(b)|\}$ such that

$$\delta_\star < \frac{\mathrm{Re}(b)\mathrm{Im}(a) - \mathrm{Re}(a)\mathrm{Im}(b) + \gamma\mathrm{Im}(b)}{\mathrm{Im}(a)}, \text{ in Case 3.}$$

This ends the proof. $\square$

Corollary 4.6. *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $V \in L^\infty(\Omega;\mathbb{R})$, let $0 < m < 1$ and let $(a,b,c) \in \mathbb{C}^3$ be such that $\mathrm{Im}(a) \leqslant 0$, $\mathrm{Im}(b) < 0$ and $\mathrm{Im}(c) \leqslant 0$. If $\mathrm{Re}(a) \leqslant 0$ then assume further that $\mathrm{Im}(a) < 0$. Let $\delta \geqslant 0$. Let $(F_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega) \cap L^2(\Omega)$ be bounded in $L^2(\Omega)$ and let $(u_\ell^n)_{(n,\ell)\in\mathbb{N}^2} \subset H^1(\Omega) \cap L^{m+1}(\Omega)$ be a sequence satisfying*

$$\forall n \in \mathbb{N},\ \forall \ell \in \mathbb{N},\ -\Delta u_\ell^n + \delta u_\ell^n + f_\ell\big(u_\ell^n\big) = F_n, \text{ in } L^2(\Omega), \tag{4.21}$$

with boundary condition (1.3) or (1.4), where for any $\ell \in \mathbb{N}$,

$$\forall u \in L^2(\Omega),\ f_\ell(u) = \begin{cases} a|u|^{-(1-m)}u + (b-\delta)u + cV^2u, & \text{if } |u| \leqslant \ell, \\ a\ell^m \dfrac{u}{|u|} + (b-\delta)\ell\dfrac{u}{|u|} + cV^2\ell\dfrac{u}{|u|}, & \text{if } |u| > \ell. \end{cases} \tag{4.22}$$

For (1.4), Ω is assumed to have a C^1 boundary. Then there exist $M = M\big(\|V\|_{L^\infty(\Omega)}, |a|, |b|, |c|\big)$ and a diagonal extraction $\Big(u_{\varphi(n)}^n\Big)_{n\in\mathbb{N}}$ of $(u_\ell^n)_{(n,\ell)\in\mathbb{N}^2}$ for which,

$$\begin{aligned} \big\|\nabla u_{\varphi(n)}^n\big\|_{L^2(\Omega)}^2 + \big\|u_{\varphi(n)}^n\big\|_{L^2\left(\left\{\left|u_{\varphi(n)}^n\right|\leqslant\varphi(n)\right\}\right)}^2 + \big\|u_{\varphi(n)}^n\big\|_{L^{m+1}\left(\left\{\left|u_{\varphi(n)}^n\right|\leqslant\varphi(n)\right\}\right)}^{m+1} \\ + \big\|u_{\varphi(n)}^n\big\|_{L^1\left(\left\{\left|u_{\varphi(n)}^n\right|>\varphi(n)\right\}\right)} \leqslant M \sup_{n\in\mathbb{N}} \|F_n\|_{L^2(\Omega)}^2, \end{aligned}$$

for any $n \in \mathbb{N}$. The same is true if we replace the conditions on (a,b,c) by $(a,b,c) \in \mathbb{A} \times \mathbb{A} \times \{0\}$ satisfies (2.15) and $\delta \leqslant \delta_\star$, where $\delta_\star$ is given by Lemma 4.5. In this case, $M = M(|a|,|b|)$.

Proof. One easily sees that $(f_\ell)_{\ell\in\mathbb{N}} \subset C\big(L^2(\Omega) \cap L^{m+1}(\Omega); L^2(\Omega) + L^{\frac{m+1}{m}}(\Omega)\big)$ and it follows that u_ℓ^n and $\mathrm{i}u_\ell^n$ are admissible test functions in (2.10). We then obtain,

$$\begin{aligned} \|\nabla u_\ell^n\|_{L^2(\Omega)}^2 + \mathrm{Re}(a)\left(\|u_\ell^n\|_{L^{m+1}(\{|u_\ell^n|\leqslant\ell\})}^{m+1} + \ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) \\ + \big(\mathrm{Re}(b) - \|V\|_{L^\infty(\Omega)}^2|\mathrm{Re}(c)|\big)\left(\|u_\ell^n\|_{L^2(\{|u_\ell^n|\leqslant\ell\})}^2 + \ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) \leqslant \mathrm{Re}\int_\Omega F_n\overline{u_\ell^n}\mathrm{d}x, \end{aligned} \tag{4.23}$$

$$\begin{aligned} \mathrm{Im}(a)\left(\|u_\ell^n\|_{L^{m+1}(\{|u_\ell^n|\leqslant\ell\})}^{m+1} + \ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) + \mathrm{Im}(b)\left(\|u_\ell^n\|_{L^2(\{|u_\ell^n|\leqslant\ell\})}^2 + \ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) \\ + \mathrm{Im}(c)\left(\|Vu\|_{L^2(\{|u_\ell^n|\leqslant\ell\}))}^2 + \ell\|V^2u\|_{L^1(\{|u_\ell^n|>\ell\}))}\right) = \mathrm{Im}\int_\Omega F_n\overline{u_\ell^n}\mathrm{d}x, \end{aligned} \tag{4.24}$$

for any $(n,\ell)\in\mathbb{N}^2$. If $(a,b,c)\in\mathbb{A}\times\mathbb{A}\times\{0\}$ satisfies (2.15), then we obtain

$$\|\nabla u_\ell^n\|_{L^2(\Omega)}^2+\delta\|u_\ell^n\|_{L^2(\Omega)}^2+\mathrm{Re}(a)\left(\|u_\ell^n\|_{L^{m+1}(\{|u_\ell^n|\leqslant\ell\})}^{m+1}+\ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right)\\+\big(\mathrm{Re}(b)-\delta\big)\left(\|u_\ell^n\|_{L^2(\{|u_\ell^n|\leqslant\ell\})}^2+\ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right)=\mathrm{Re}\int_\Omega F_n\overline{u_\ell^n}\mathrm{d}x, \tag{4.25}$$

$$\mathrm{Im}(a)\left(\|u_\ell^n\|_{L^{m+1}(\{|u_\ell^n|\leqslant\ell\})}^{m+1}+\ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right)\\+\mathrm{Im}(b)\left(\|u_\ell^n\|_{L^2(\{|u_\ell^n|\leqslant\ell\})}^2+\ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right)=\mathrm{Im}\int_\Omega F_n\overline{u_\ell^n}\mathrm{d}x, \tag{4.26}$$

for any $(n,\ell)\in\mathbb{N}^2$. For this last case, it follows from Lemma 4.5, Hölder's inequality and Young's inequality (3.6) applied with $x=|F|$, $y=|u_\ell^n|$ and $\mu=\sqrt{\frac{M}{L}}$ that

$$\|\nabla u_\ell^n\|_{L^2(\Omega)}^2+\frac{L}{2}\|u_\ell^n\|_{L^2(\{|u_\ell^n|\leqslant\ell\})}^2+L\big\|u_\ell^n\big\|_{L^{m+1}}^{m+1}\big(\{|u_\ell^n|\leqslant\ell\}\big)\\+\big(L\ell-M\|F\|_{L^\infty(\Omega)}\big)\,\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\leqslant\frac{M^2}{2L}\|F\|_{L^2(\Omega)}^2.$$

Then the result follows by choosing for each $n\in\mathbb{N}$, $\varphi(n)\in\mathbb{N}$ large enough to have $L\varphi(n)-M\|F\|_{L^\infty(\Omega)}\geqslant 1$. Now we turn out to the case (4.23)–(4.24). Let M and A be given by Lemma 4.4 with $R=\|V\|_{L^\infty(\Omega)}$. For each $n\in\mathbb{N}$, let $\varphi(n)\in\mathbb{N}$ be large enough to have $\varphi(n)\geqslant 2A\|F_n\|_{L^\infty(\Omega)}+1$, if $|\omega|<|\Omega|$ and $\varphi(n)=n$, if $|\omega|=|\Omega|$. For each $n\in\mathbb{N}$, with help of (4.23) and (4.24), we may apply Lemma 4.4 to $u_{\varphi(n)}^n$ with $\omega=\left\{x\in\Omega;\left|u_{\varphi(n)}^n(x)\right|\leqslant\varphi(n)\right\}$, $\alpha=\varphi(n)^m$, $\beta=\varphi(n)$ and $R=\|V\|_{L^\infty(\Omega)}$. Hence the result. □

5 Proofs of the main results

Proof of Theorem 2.15. Let the assumptions of the theorem be fulfilled. Setting $f(u)=a|u|^{-(1-m)}u+Vu$, we easily check that,

$$f\in C\big(L_{\mathrm{loc}}^{s+\varepsilon}(\Omega);L_{\mathrm{loc}}^s(\Omega)\big)\cap C\big(L^{s+\varepsilon}(\Omega);L^s(\Omega)\big),$$

for any $1<s<\infty$ and $0<\varepsilon<s-1$. Then the notions of local and global very weak solution make sense and by Remark 2.16, boundary conditions in Properties 2) and 3) make sense. Property 1) follows from Proposition 4.5 in Bégout and Díaz [3] while Property 2) comes from Remark 4.7 in Bégout and Díaz [3]. It remains to establish Property 3). Assume first that $F\in L^p(\Omega)$ and $V\in\bigcap\limits_{1<r<\infty}L^r(\Omega)$. It follows from the equation that for any $\varepsilon\in(0,q-1)$, $\Delta u\in L^{q-\varepsilon}(\Omega)$. We now recall an elliptic regularity result. If for some $1<s<\infty$, $u\in L^s(\Omega)$ satisfies $\Delta u\in L^s(\Omega)$ and $\gamma(\nabla u.\nu)=0$ then $u\in W^{2,s}(\Omega)$

(Proposition 2.5.2.3, p.131, in Grisvard [12]). Since for any $\varepsilon \in (0, q-1)$, $u, \Delta u \in L^{q-\varepsilon}(\Omega)$ and $\gamma(\nabla u.\nu) = 0$ (by assumption), by following the bootstrap method of the proof p.52 of Property 1) of Proposition 4.5 in Bégout and Díaz [3], we obtain the result. Indeed, therein, it is sufficient to apply the global regularity result in Grisvard [12] (Proposition 2.5.2.3, p.131) in place of the local regularity result in Cazenave [7] (Proposition 4.1.2, p.101-102). Now, you turn out to the Hölder regularity. Assume $F \in C^{0,\alpha}(\overline{\Omega})$ and $V \in C^{0,\alpha}(\overline{\Omega})$. By global smoothness property in $W^{2,p}$ proved above, we know that $u \in W^{2,N+1}(\Omega)$ and $\gamma(\nabla u.\nu) = 0$ in $L^{N+1}(\Gamma)$. It follows from the Sobolev's embedding, $W^{2,N+1}(\Omega) \hookrightarrow C^{1,\frac{1}{N+1}}(\overline{\Omega}) \hookrightarrow C^{0,1}(\overline{\Omega})$, that for any $x \in \Gamma$, $\frac{\partial u}{\partial \nu}(x) = 0$ and $u \in C^{0,1}(\overline{\Omega})$. A straightforward calculation yields,

$$\forall (x,y) \in \overline{\Omega}^2, \ \left| |u(x)|^{-(1-m)}u(x) - |u(y)|^{-(1-m)}u(y) \right| \leqslant 5|u(x) - u(y)|^m \leqslant 5|x-y|^m.$$

Setting, $g = F - (a|u|^{-(1-m)}u + (b-1)u + cVu)$, we deduce that $g \in C^{0,\alpha}(\overline{\Omega})$. Let $v \in C^{2,\alpha}(\overline{\Omega})$ be the unique solution to

$$\begin{cases} -\Delta v + v = g, & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0, & \text{on } \Gamma, \end{cases}$$

(see, for instance, Theorem 3.2 p.137 in Ladyzhenskaya and Ural'tseva [15]). It follows that u and v are two H^1 weak solutions of the above equations (in the sense of Definition 2.1), and since uniqueness holds in $H^1(\Omega)$ (Lax-Milgram's Theorem), we deduce that $u = v$. Hence $u \in C^{2,\alpha}(\overline{\Omega})$. This concludes the proof[4]. □

Proof of Proposition 2.17. Let the assumptions of the proposition be fulfilled. Setting $f(u) = a|u|^{-(1-m)}u + Vu$,it follows from Sobolev's embedding that,

$$f \in C\big(H^1(\Omega); L^2(\Omega)\big).$$

We first establish Property 1). Since Ω has $C^{0,1}$ boundary and $u \in H^1_0(\Omega)$, it follows that $\gamma(u) = 0$. Moreover, Sobolev's embedding and equation (2.17) implies that $\Delta u \in L^2(\Omega)$. We then obtain that $u \in H^2(\Omega)$ (Grisvard [12], Corollary 2.5.2.2 p.131). Hence Property 1). We turn out to Property 2). It follows from equation (2.17) that $\Delta u \in L^2(\Omega)$, so that (2.17) takes sense almost everywhere in Ω. Then Property 2) comes from the arguments of 3) of Remark 2.2. □

Lemma 5.1. *Let $\mathcal{O} \subset \mathbb{R}^N$ be a nonempty bounded open subset, let $V \in L^\infty(\Omega; \mathbb{C})$, let $0 < m < 1$, let $(a,b,c) \in \mathbb{C}^3$ and let $F \in L^2(\mathcal{O})$. Let $\delta \in [0,1]$. Then for any $\ell \in \mathbb{N}$, there exist a weak solution*

[4] More directly, we could have said that since $u \in W^{2,N+1}(\Omega)$, $\gamma(\nabla u.\nu) = 0$ and $\Delta u \in C^{0,\alpha}(\overline{\Omega})$ (by the estimate of the nonlinearity) then by Theorem 6.3.2.1, p.287, in Grisvard [12], $u \in C^{2,\alpha}(\overline{\Omega})$. But this theorem requires Ω to have a $C^{2,1}$ boundary.

$u_\ell^1 \in H_0^1(\mathcal{O})$ to

$$-\Delta u_\ell + \delta u_\ell + f_\ell(u_\ell) = F, \text{ in } L^2(\mathcal{O}), \tag{5.1}$$

with boundary condition (1.3) and a weak solution $u_\ell^2 \in H^1(\mathcal{O})$ to (5.1) with boundary condition (1.4) (in this case, $\mathcal{O}$ is assumed to have a C^1 boundary and $\delta > 0$) in the sense of Definition 2.1, where

$$\forall u \in L^2(\Omega),\ f_\ell(u) = \begin{cases} a|u|^{-(1-m)}u + (b-\delta)u + cV^2u, & \text{if } |u| \leqslant \ell, \\ a\ell^m \dfrac{u}{|u|} + (b-\delta)\ell\dfrac{u}{|u|} + cV^2\ell\dfrac{u}{|u|}, & \text{if } |u| > \ell. \end{cases} \tag{5.2}$$

If furthermore for any $\mathcal{R} \in SO_N(\mathbb{R})$, $\mathcal{R}\mathcal{O} = \mathcal{O}$ and if F is spherically symmetric then there exists a weak solution to (5.1) which is also spherically symmetric. For $N = 1$, this means that if F is an even (respectively, odd) function then u is also even (respectively, an odd) function.

Proof. We proceed with the proof in two steps. Let $H = H_0^1(\mathcal{O})$, in the homogeneous Dirichlet case, and $H = H^1(\mathcal{O})$, in the homogeneous Neumann case. Let $\delta \in [0,1]$ $\big($with additionally $\delta > 0$ and Γ of class C^1 if $H = H^1(\mathcal{O})\big)$.

Step 1. For any $G \in L^2(\mathcal{O})$, there exists a unique weak solution $u \in H$ to $-\Delta u + \delta u = G$, in the sense of Definition 2.1. Moreover, there exists $\alpha > 0$ such that for any $G \in L^2(\mathcal{O})$, $\big\|(-\Delta + \delta I)^{-1}G\big\|_{H^1(\mathcal{O})} \leqslant \alpha\|G\|_{L^2(\mathcal{O})}$. If furthermore for any $\mathcal{R} \in SO_N(\mathbb{R})$, $\mathcal{R}\mathcal{O} = \mathcal{O}$ and if G is spherically symmetric then the weak solution is also spherically symmetric. For $N = 1$, this means that if G is an even (respectively, odd) function then u is also an even (respectively, odd) function.

Step 1 is obvious. For existence and uniqueness, apply Lax-Milgram's Theorem and Poincaré's inequality (3.2), in the homogeneous Dirichlet case, and Lax-Milgram's Theorem, in the homogeneous Neumann case. The constant α comes from Poincaré's inequality and Cauchy-Schwarz's inequality, in the homogeneous Dirichlet case, and is equal to $\frac{1}{\delta}$ (still by Cauchy-Schwarz's inequality), in the homogeneous Neumann case. Spherically property follows by working in H_{rad} (the space of functions $f \in H$ such that f is spherically symmetric) in place of H $\big($and in $H_{\mathrm{odd}} \overset{\mathrm{def}}{=} \big\{v \in H;\ v \text{ is odd}\big\}$ in place of H in the odd case when $N = 1\big)$.

Step 2. Conclusion.

For each $\ell \in \mathbb{N}$, we define $g_\ell = -f_\ell + F \in C\big(L^2(\mathcal{O}); L^2(\mathcal{O})\big)$. With help of the continuous and compact embedding $i : H \hookrightarrow L^2(\mathcal{O})$ and Step 1, we may define a continuous and compact sequence of mappings $(T_\ell)_{\ell\in\mathbb{N}}$ of H as follows. For any $\ell \in \mathbb{N}$, set

$$\begin{array}{cccccccc} T_\ell : H & \overset{i}{\hookrightarrow} & L^2(\mathcal{O}) & \overset{g_\ell}{\longrightarrow} & L^2(\mathcal{O}) & \xrightarrow{(-\Delta+\delta I)^{-1}} & H \\ u & \longmapsto & i(u) = u & \longmapsto & g_\ell(u) & \longmapsto & (-\Delta + \delta u)^{-1}(g_\ell)(u) \end{array}$$

Let $\ell\in\mathbb{N}$. Set $\rho=2\alpha(|a|+|b|+|c|+1)\left(\left(\|V\|_{L^\infty(\Omega)}^2+2\right)\ell|\mathcal{O}|^{\frac12}+\|F\|_{L^2(\mathcal{O})}\right)$. Let $u\in H$. It follows that,

$$\|T_\ell(u)\|_{H^1(\mathcal{O})}=\left\|(-\Delta+\delta I)^{-1}(g_\ell)(u)\right\|_{H^1(\mathcal{O})}\leqslant\alpha\|g_\ell(u)\|_{L^2(\mathcal{O})}$$
$$\leqslant 2\alpha(|a|+|b|+|c|+1)\left(\left(\ell^m+\ell+\ell\|V\|_{L^\infty(\Omega)}^2\right)|\mathcal{O}|^{\frac12}+\|F\|_{L^2(\mathcal{O})}\right)\leqslant\rho.$$

Hence, $T_\ell(H)\subset\overline{B}_H(0,\rho)$, where $\overline{B}_H(0,\rho)=\left\{u\in H;\|u\|_{H^1(\mathcal{O})}\leqslant\rho\right\}$. Existence then comes from the Schauder's fixed point Theorem applied to T_ℓ. We obviously obtain the spherically symmetry property by working in the functional spaces H_{rad} in place of H $\big($and in H_{odd} in place of H in the odd case when $N=1\big)$. □

Proof of Theorem 2.4. Let for any $u\in L^2(\Omega)$, $f(u)=a|u|^{-(1-m)}u+bu$. For any $n\in\mathbb{N}_0$, we write $\Omega_n=\Omega\cap B(0,n_0+n)$, where $n_0\in\mathbb{N}$ is large enough to have $\Omega_0\neq\emptyset$. Let $(G_n)_{n\in\mathbb{N}}\subset\mathscr{D}(\Omega)$ be such that

$$G_n\xrightarrow[n\to\infty]{L^2(\Omega)}F. \tag{5.3}$$

Let $\left(u_\ell^n\right)_{(n,\ell)\in\mathbb{N}^2}\subset H_0^1(\Omega_n)$ a sequence of weak solutions of (5.1) be given by Lemma 5.1 with $\mathcal{O}=\Omega_n$, $c=\delta=0$ and $F_n=G_{n|\Omega_n}$. We define $\widetilde{u_\ell^n}\in H_0^1(\Omega)$ by extending u_n by 0 in $\Omega\cap\Omega_n^{\mathrm{c}}$. We also denote by $\widetilde{f_\ell}$ the extension by 0 of f_ℓ in $\Omega\cap\Omega_n^{\mathrm{c}}$. By Corollary 4.3, there exists a diagonal extraction $\left(\widetilde{u_{\varphi(n)}^n}\right)_{n\in\mathbb{N}}$ of $\left(\widetilde{u_\ell^n}\right)_{(n,\ell)\in\mathbb{N}^2}$ which is bounded in $H_0^1(\Omega)$. By reflexivity of $H_0^1(\Omega)$, Rellich-Kondrachov's Theorem and converse of the dominated convergence theorem, there exist $u\in H_0^1(\Omega)$ and $g\in L^2_{\mathrm{loc}}(\Omega;\mathbb{R})$ such that, up to a subsequence that we still denote by $\left(\widetilde{u_{\varphi(n)}^n}\right)_{n\in\mathbb{N}}$,

$$\begin{aligned}
&\widetilde{u_{\varphi(n)}^n}\xrightarrow[n\to\infty]{L^2_{\mathrm{loc}}(\Omega)}u,\\
&\widetilde{u_{\varphi(n)}^n}\xrightarrow[n\to\infty]{\text{a.e. in }\Omega}u,\\
&\left|\widetilde{u_{\varphi(n)}^n}\right|\leqslant g,\ \text{for any } n\in\mathbb{N},\ \text{a.e. in }\Omega,
\end{aligned}\tag{5.4}$$

These two last estimates yield,

$$\begin{aligned}
&\widetilde{f_{\varphi(n)}}\left(\widetilde{u_{\varphi(n)}^n}\right)\xrightarrow[n\to\infty]{\text{a.e. in }\Omega}f(u),\\
&\forall n\in\mathbb{N},\ \left|\widetilde{f_{\varphi(n)}}\left(\widetilde{u_{\varphi(n)}^n}\right)\right|\leqslant C(g^m+g)\in L^2_{\mathrm{loc}}(\Omega),\ \text{a.e. in }\Omega.
\end{aligned}$$

It follows from the dominated convergence Theorem that

$$\widetilde{f_{\varphi(n)}}\left(\widetilde{u_{\varphi(n)}^n}\right)\xrightarrow[n\to\infty]{L^2_{\mathrm{loc}}(\Omega)}f(u). \tag{5.5}$$

Let $\varphi \in \mathscr{D}(\Omega)$. Let $n_\star \in \mathbb{N}$ be large enough to have $\operatorname{supp}\varphi \subset \Omega_{n_\star}$. We have by (5.1) and Definition 2.1,

$$\forall n > n_\star,\ \left\langle -\mathrm{i}\Delta u^n_{\varphi(n)} + f_{\varphi(n)}\left(u^n_{\varphi(n)}\right) - F_n, \varphi_{|\Omega_n}\right\rangle_{\mathscr{D}'(\Omega_n),\mathscr{D}(\Omega_n)} = 0. \tag{5.6}$$

Estimates (5.4), (5.5), (5.3) and (5.6) lead to,

$$\begin{aligned}
&\langle -\Delta u + f(u) - F, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} \\
&= \langle -u, \Delta\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} + \langle f(u) - F, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} \\
&= \lim_{n\to\infty}\left\langle -\widetilde{u^n_{\varphi(n)}}, \Delta\varphi\right\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} + \lim_{n\to\infty}\left\langle \widetilde{f_{\varphi(n)}\left(u^n_{\varphi(n)}\right)} - G_n, \varphi\right\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} \\
&= \lim_{n\to\infty}\left\langle -u^n_{\varphi(n)}, \Delta\left(\varphi_{|\Omega_n}\right)\right\rangle_{\mathscr{D}'(\Omega_n),\mathscr{D}(\Omega_n)} + \lim_{n\to\infty}\left\langle f_{\varphi(n)}\left(u^n_{\varphi(n)}\right) - G_{n|\Omega_n}, \varphi_{|\Omega_n}\right\rangle_{\mathscr{D}'(\Omega_n),\mathscr{D}(\Omega_n)} \\
&= \lim_{n\to\infty}\left\langle -\Delta u^n_{\varphi(n)} + f_{\varphi(n)}\left(u^n_{\varphi(n)}\right) - F_n, \varphi_{|\Omega_n}\right\rangle_{\mathscr{D}'(\Omega_n),\mathscr{D}(\Omega_n)} \\
&= 0.
\end{aligned}$$

By density, we then obtain that $u \in H^1_0(\Omega)$ satisfies (2.2) for any $v \in H^1_0(\Omega)$, so that $u \in H^1_0(\Omega)$ is a weak solution to

$$-\Delta u + f(u) = F, \text{ in } L^2(\Omega).$$

Furthermore, any weak solution belongs to $H^2_{\mathrm{loc}}(\Omega)$ (Theorem 2.15). Finally, if F is spherically symmetric then u (obtained as a limit of weak solutions given by Lemma 5.1) is also spherically symmetric. For $N = 1$, this includes the case where F is an even function. $\square$

Proof of Theorems 2.5 and 2.11 . Set for any $u \in L^2(\Omega) \cap L^{m+1}(\Omega)$, $f(u) = a|u|^{-(1-m)}u + bu$. It follows from Example 2.3 that u and $\mathrm{i}u$ are admissible test functions in (2.10). We then obtain,

$$\|\nabla u\|^2_{L^2(\Omega)} + \mathrm{Re}(a)\|u\|^{m+1}_{L^{m+1}(\Omega)} + \mathrm{Re}(b)\|u\|^2_{L^2(\Omega)} = \mathrm{Re}\int_\Omega F\overline{u}\mathrm{d}x,$$
$$\mathrm{Im}(a)\|u\|^{m+1}_{L^{m+1}(\Omega)} + \mathrm{Im}(b)\|u\|^2_{L^2(\Omega)} = \mathrm{Im}\int_\Omega F\overline{u}\mathrm{d}x.$$

Theorem 2.5 follows immediately from Lemma 4.1 applied with $\omega = \Omega$ and Remark 4.2, while Theorem 2.11 is a consequence of Lemma 4.5 applied with $\delta = 0$ and Young's inequality 3.6. This ends the proof. $\square$

Proof of Theorem 2.7. By Example 2.3, u and $\mathrm{i}u$ are admissible test functions in (2.8). We then obtain,

$$\|\nabla u\|^2_{L^2(\Omega)} + \mathrm{Re}(a)\|u\|^{m+1}_{L^{m+1}(\Omega)} + \left(\mathrm{Re}(b) - |\mathrm{Re}(c)|\|V\|^2_{L^\infty(\Omega)}\right)\|u\|^2_{L^2(\Omega)} \leqslant \int_\Omega |Fu|\mathrm{d}x,$$
$$|\mathrm{Im}(a)|\|u\|^{m+1}_{L^{m+1}(\Omega)} + |\mathrm{Im}(b)|\|u\|^2_{L^2(\Omega)} + |\mathrm{Im}(c)|\|Vu\|^2_{L^2(\Omega)} \leqslant \int_\Omega |Fu|\mathrm{d}x.$$

The theorem follows Lemma 4.4 applied with $\omega=\Omega$, $R=\|V\|_{L^\infty(\Omega)}$ and $\alpha=\beta=0$. $\square$

Proof of Theorems 2.6 and 2.9. Let the assumptions of Theorems 2.6 and 2.9 be fulfilled.

Proof of the existence. We first assume that Ω is bounded. Let $H=H^1_0(\Omega)$, in the homogeneous Dirichlet case, and $H=H^1(\Omega)$, in the homogeneous Neumann case. Let $\delta_\star$ be given by Lemma 4.5 and let for any $u\in L^2(\Omega)$, $f(u)=a|u|^{-(1-m)}u+bu+cV^2u$ (with $c=0$ in the case of Theorem 2.9). Let $(F_n)_{n\in\mathbb{N}}\subset\mathscr{D}(\Omega)$ be such that

$$F_n\xrightarrow[n\to\infty]{L^2(\Omega)}F. \tag{5.7}$$

Let $\left(u^n_\ell\right)_{(n,\ell)\in\mathbb{N}^2}\subset H$ a sequence of weak solutions of (5.1) be given by Lemma 5.1 with $\mathcal{O}=\Omega$, $\delta=1$ for Theorem 2.6, $\delta=\delta_\star$ for Theorem 2.9 and such F_n. By Corollary 4.6, there exists a diagonal extraction $\left(u^n_{\varphi(n)}\right)_{n\in\mathbb{N}}$ of $\left(u^n_\ell\right)_{(n,\ell)\in\mathbb{N}^2}$ which is bounded in $W^{1,1}(\Omega)\cap\dot H^1(\Omega)$. Let $1<p<2$ be such that $W^{1,1}(\Omega)\hookrightarrow L^p(\Omega)$. Then $\left(u^n_{\varphi(n)}\right)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$ and there exist $u\in W^{1,p}(\Omega)\cap\dot H^1(\Omega)$ and $g\in L^p(\Omega;\mathbb{R})$ such that, up to a subsequence that we still denote by $\left(u^n_{\varphi(n)}\right)_{n\in\mathbb{N}}$,

$$u^n_{\varphi(n)}\xrightarrow[n\to\infty]{L^p(\Omega)}u, \tag{5.8}$$

$$\nabla u^n_{\varphi(n)}\rightharpoonup\nabla u\text{ in }\left(L^2_{\mathrm{w}}(\Omega)\right)^N,\text{ as }n\longrightarrow\infty, \tag{5.9}$$

$$u^n_{\varphi(n)}\xrightarrow[n\to\infty]{\text{a.e. in }\Omega}u,$$

$$\left|u^n_{\varphi(n)}\right|\leqslant g,\ \text{for any }n\in\mathbb{N},\ \text{a.e. in }\Omega,$$

$$\left(u^n_{\varphi(n)}\mathbb{1}_{\left\{\left|u^n_{\varphi(n)}\right|\leqslant\varphi(n)\right\}}\right)_{n\in\mathbb{N}}\text{ is bounded in }L^2(\Omega),$$

where the last estimate comes from Corollary 4.6. These three last estimates and Fatou's Lemma yield,

$$\begin{aligned}
&u\in L^2(\Omega),\\
&f_{\varphi(n)}\left(u^n_{\varphi(n)}\right)\xrightarrow[n\to\infty]{\text{a.e. in }\Omega}f(u)-\delta u,\\
&\forall n\in\mathbb{N},\ \left|f_{\varphi(n)}\left(u^n_{\varphi(n)}\right)\right|\leqslant C(g^m+g)\in L^p(\Omega),\ \text{a.e. in }\Omega.
\end{aligned}$$

It follows that $u\in H^1(\Omega)$. From the dominated convergence Theorem, we get

$$f_{\varphi(n)}\left(u^n_{\varphi(n)}\right)\xrightarrow[n\to\infty]{L^p(\Omega)}f(u)-\delta u. \tag{5.10}$$

We claim that in the case of Dirichlet boundary condition, one has $u\in H^1_0(\Omega)$. We recall a Gagliardo-Nirenberg's inequality.

$$\forall w\in H^1_0(\Omega),\ \|w\|^{N+2}_{L^2(\Omega)}\leqslant C\|w\|^2_{L^1(\Omega)}\|\nabla w\|^N_{L^2(\Omega)},$$

where $C=C(N)$. In particular, C does not depend on Ω. Since $\left(u^n_{\varphi(n)}\right)_{n\in\mathbb{N}}\subset H^1_0(\Omega)$ is bounded in $W^{1,1}(\Omega)\cap\dot H^1(\Omega)$, it follow from the above Gagliardo-Nirenberg's inequality that $\left(u^n_{\varphi(n)}\right)_{n\in\mathbb{N}}$ is bounded in $H^1_0(\Omega)$. So that $u\in H^1_0(\Omega)$ and the claim is proved. Now, we show that $u\in H$ is a weak solution. Let $m_0\in\mathbb{N}$ be large enough to have $H^{m_0}(\Omega)\hookrightarrow L^{p'}(\Omega)$. Let $v\in\mathscr{D}(\Omega)$, if $H=H^1_0(\Omega)$ and let $v\in H^{m_0}(\Omega)$, if $H=H^1(\Omega)$. By (5.1), we have for any $n\in\mathbb{N}$,

$$\left\langle\nabla u^n_{\varphi(n)},\nabla v\right\rangle_{L^2(\Omega),L^2(\Omega)}+\left\langle\delta u^n_{\varphi(n)}+f_{\varphi(n)}\left(u^n_{\varphi(n)}\right),v\right\rangle_{L^p(\Omega),L^{p'}(\Omega)} - \langle F_n,v\rangle_{L^2(\Omega),L^2(\Omega)}=0. \quad (5.11)$$

Estimates (5.9), (5.8), (5.10), (5.7) and (5.11) lead to,

$$\begin{aligned}
&\langle\nabla u,\nabla v\rangle_{L^2(\Omega),L^2(\Omega)}+\langle f(u)-F,v\rangle_{L^2(\Omega),L^2(\Omega)}\\
&=\langle\nabla u,\nabla v\rangle_{L^2(\Omega),L^2(\Omega)}+\left\langle\delta u+\big(f(u)-\delta u\big),v\right\rangle_{L^p(\Omega),L^{p'}(\Omega)}-\langle F,v\rangle_{L^2(\Omega),L^2(\Omega)}\\
&=\lim_{n\to\infty}\left\langle\nabla u^n_{\varphi(n)},\nabla v\right\rangle_{L^2,L^2}+\lim_{n\to\infty}\left\langle\delta u^n_{\varphi(n)}+f_{\varphi(n)}\left(u^n_{\varphi(n)}\right),v\right\rangle_{L^p,L^{p'}}-\lim_{n\to\infty}\langle F_n,v\rangle_{L^2,L^2}\\
&=\lim_{n\to\infty}\left(\left\langle\nabla u^n_{\varphi(n)},\nabla v\right\rangle_{L^2,L^2}+\left\langle\delta u^n_{\varphi(n)}+f_{\varphi(n)}\left(u^n_{\varphi(n)}\right),v\right\rangle_{L^p,L^{p'}}-\langle F_n,v\rangle_{L^2,L^2}\right)\\
&=0.
\end{aligned}$$

By density of $\mathscr{D}(\Omega)$ in $H^1_0(\Omega)$ and density of $H^{m_0}(\Omega)$ in $H^1(\Omega)$ (see, for instance, Corollary 9.8, p.277, in Brezis [4]), it follows that

$$\forall v\in H,\ \langle\nabla u,\nabla v\rangle_{L^2(\Omega),L^2(\Omega)}+\langle f(u),v\rangle_{L^2(\Omega),L^2(\Omega)}=\langle F,v\rangle_{L^2(\Omega),L^2(\Omega)}.$$

Finally, $u\in H^2_{\mathrm{loc}}(\Omega)$ (Theorem 2.15). This finishes the proof of the existence when Ω is bounded. Assume Ω is arbitrary. Then we want to solve (1.2) and (1.3). Let $n_0\in\mathbb{N}$ ve such that $\Omega\cap B(0,n_0)\neq\emptyset$ and set for any $n\in\mathbb{N}$, $\Omega_n=\Omega\cap B(0,n+n_0)$. It follows from the above proof that for any $n\in\mathbb{N}$, equations (1.1), with external source $F_n=F_{|\Omega_n}$, and (1.3) admit at least one weak solution $u\in H^1_0(\Omega_n)$. Extending u_n by 0 in $\Omega\cap\Omega_n^{\mathrm{c}}$ and denoting by U_n this extension, we get by Theorems 2.7 and 2.11 that $(U_n)_{n\in\mathbb{N}}$ is bounded in $H^1_0(\Omega)\cap L^{m+1}(\Omega)$. Then, up to a subsequence that we still denote by $(U_n)_{n\in\mathbb{N}}$, there exist $u\in H^1_0(\Omega)\cap L^{m+1}(\Omega)$ and $g\in L^2_{\mathrm{loc}}(\Omega;\mathbb{R})$ such that $U_n\rightharpoonup u$ in $H^1_{\mathrm{w}}(\Omega)$, as $n\to\infty$, $U_n\xrightarrow[n\to\infty]{L^2_{\mathrm{loc}}(\Omega)}u$, $U_n\xrightarrow[n\to\infty]{\text{a.e. in }\Omega}u$ and $|U_n|\leqslant g$, a.e. in Ω. In particular, $|U_n|^m\leqslant g^m\in L^{\frac{m+1}{m}}_{\mathrm{loc}}(\Omega)$, a.e. in Ω, so that $|U_n|^{-(1-m)}U_n\xrightarrow[n\to\infty]{L^{\frac{m+1}{m}}_{\mathrm{loc}}(\Omega)}|u|^{-(1-m)}u$. Let $\varphi\in\mathscr{D}(\Omega)$. Let $n_\star\in\mathbb{N}$ be such that $\operatorname{supp}\varphi\subset\Omega_{n_\star}$. It follows that for each $n>n_\star$, U_n satisfies (2.8) in Ω with $F=F_n$ and $v=\varphi$. The above convergencies for $(U_n)_{n\in\mathbb{N}}$ allow to pass in the limit in (2.8) and, by density of $\mathscr{D}(\Omega)$ in $H^1_0(\Omega)\cap L^{m+1}(\Omega)$, we get that $u\in H^1_0(\Omega)\cap L^{m+1}(\Omega)$ satisfies (2.8) for any $v\in H^1_0(\Omega)\cap L^{m+1}(\Omega)$. So that u is a weak

solution to (1.2) and (1.3). Furthermore, by Theorem 2.15, any weak solution belongs to $H^2_{\mathrm{loc}}(\Omega)$. This achieves the proof of existence.

Proof of symmetry property. If F is spherically symmetric then u (obtained as a limit of weak solutions given by Lemma 5.1) is also spherically symmetric and belongs to $H^2_{\mathrm{loc}}(\Omega)$ (Theorem 2.15). For $N=1$, this includes the case where F is an even function. This concludes the proof of the theorems. $\square$

Proof of Theorem 2.12. Let $u_1,u_2\in H^1(\Omega)\cap L^{m+1}(\Omega)$ be two weak solutions of (1.2). Set

$$\Sigma=\left\{v\in L^2(\Omega);Vv\in L^2(\Omega)\right\},\ u=u_1-u_2,\ f(v)=|v|^{-(1-m)}v,$$

for any $v\in L^{m+1}(\Omega)$, and let for any $v\in\Sigma\cap L^{m+1}(\Omega)$,

$$g(v)=af(v)+bv+cV^2v.$$

Assume that $Vu_1,Vu_2\in\Sigma$. By 1) of Example 2.3, we deduce that u satisfies

$$-\Delta u+g(u_1)-g(u_2)=0,\ \text{in }\Sigma^\star+L^{\frac{m+1}{m}}(\Omega).$$

From Lemma 9.1 in Bégout and Díaz [3], there exists a positive constant C such that,

$$C\int\limits_\omega\frac{|u_1(x)-u_2(x)|^2}{(|u_1(x)|+|u_2(x)|)^{1-m}}\mathrm{d}x\leqslant\langle f(u_1)-f(u_2),u_1-u_2\rangle_{L^{\frac{m+1}{m}}(\Omega),L^{m+1}(\Omega)},\tag{5.12}$$

where $\omega=\Big\{x\in\Omega;|u_1(x)|+|u_2(x)|>0\Big\}$.

Proof of 1) of the theorem. We may choose $v=au$ in (2.8) as a test function. We then get,

$$\mathrm{Re}(a)\|\nabla u\|_{L^2}^2+|a|^2\langle f(u_1)-f(u_2),u_1-u_2\rangle_{L^{\frac{m+1}{m}},L^{m+1}}+\mathrm{Re}(a\overline{b})\|u\|_{L^2}^2+\mathrm{Re}\left(a\overline{c}\right)\|Vu\|_{L^2}^2=0.$$

It follows from the above estimate and (5.12) that,

$$\mathrm{Re}(a)\|\nabla u\|_{L^2}^2+C|a|^2\int\limits_\omega\frac{|u_1(x)-u_2(x)|^2}{(|u_1(x)|+|u_2(x)|)^{1-m}}\mathrm{d}x+\mathrm{Re}(a\overline{b})\|u\|_{L^2}^2+\mathrm{Re}\left(a\overline{c}\right)\|Vu\|_{L^2}^2\leqslant0.$$

Then 1) follows.

Proof of 2) of the theorem. Choosing $v=bu$ in (2.8) as a test function and recalling that $\mathrm{Im}(a\overline{b})=0$, we get with help of (5.12) that,

$$\mathrm{Re}(b)\|\nabla u\|_{L^2}^2+C\mathrm{Re}(a\overline{b})\int\limits_\omega\frac{|u_1(x)-u_2(x)|^2}{(|u_1(x)|+|u_2(x)|)^{1-m}}\mathrm{d}x+|b|^2\|u\|_{L^2}^2+\mathrm{Re}(b\overline{c})\|Vu\|_{L^2}^2\leqslant0.$$

Hence 2) follows.

Proof of 3) of the theorem. Choosing $v = cu$ in (2.8) as a test function and recalling that $\mathrm{Im}(a\bar{c}) = 0$, we get with help of (5.12) that,

$$\mathrm{Re}(c)\|\nabla u\|_{L^2}^2 + C\mathrm{Re}(a\bar{c}) \int\limits_{\omega} \frac{|u_1(x) - u_2(x)|^2}{(|u_1(x)| + |u_2(x)|)^{1-m}} \mathrm{d}x + \mathrm{Re}(b\bar{c})\|u\|_{L^2}^2 + |c|^2\|Vu\|_{L^2}^2 \leqslant 0.$$

Then 3) follows. This ends the proof of the theorem. □

Remark 5.2. It is not hard to adapt the above proof to find other criteria of uniqueness.

6 On the existence of solutions of the Dirichlet problem for data beyond $L^2(\Omega)$

In this section we shall indicate how some of the precedent results of this paper can be extended to some data F which are not in $L^2(\Omega)$ but in the more general Hilbert space $L^2(\Omega;\delta^\alpha)$, where $\delta(x) = \mathrm{dist}(x,\Gamma)$ and $\alpha \in (0,1)$.

In order to justify the associated notion of weak solution, we start by assuming that a function u solves equation (2.1) with the Dirichlet boundary condition (1.3), $u_{|\Gamma} = 0$, and we multiply (formally) by $\overline{v(x)}\delta(x)$, with $v \in H_0^1(\Omega;\delta^\alpha)$ $\big($the weighted Sobolev space associated to the weight $\delta^\alpha(x)\big)$, we integrate by parts (by Green's formula) and we take the real part. Then we get,

$$\mathrm{Re} \int\limits_\Omega \nabla u.\overline{\nabla v}\, \delta^\alpha \mathrm{d}x + \mathrm{Re} \int\limits_\Omega \overline{v}\, \nabla u.\nabla \delta^\alpha \mathrm{d}x + \mathrm{Re} \int\limits_\Omega f(u)\, \overline{v}\, \delta^\alpha \mathrm{d}x = \mathrm{Re} \int\limits_\Omega F\, \overline{v}\, \delta^\alpha \mathrm{d}x. \tag{6.1}$$

To give a meaning to the condition (6.1), we must assume that

$$F \in L^2(\Omega;\delta^\alpha), \tag{6.2}$$

where $\|F\|_{L^2(\Omega;\delta^\alpha)}^2 = \int_\Omega |F(x)|^2\delta^\alpha(x)\mathrm{d}x$, and to include in the definition of weak solution (Definition 2.1) the conditions

$$u \in H_0^1(\Omega;\delta^\alpha) \quad \text{and} \quad f(u) \in L^2(\Omega;\delta^\alpha). \tag{6.3}$$

The justification of the second term in (6.1) is far to be trivial and requires the use of a version of the following Hardy type inequality,

$$\int\limits_\Omega |v(x)|^2\delta^{-(2-\alpha)}(x)\mathrm{d}x \leqslant C \int\limits_\Omega |\nabla v(x)|^2\delta^\alpha(x)\mathrm{d}x, \tag{6.4}$$

which holds for some constant C independent of v, for any $v \in H_0^1(\Omega;\delta^\alpha)$ once we assume that

$$\Omega \text{ is a bounded domain of } \mathbb{R}^N \text{ with Hölder boundary} \tag{6.5}$$

(see, e.g., Kufner [14] and also Nečas [18], Drábek, Kufner and Nicolosi [11], p.34, and Opic and Kufner [19]). Notice that under (6.5), we know that $\delta \in W^{1,\infty}(\Omega)$ and so

$$\left|\int_\Omega \overline{v}\,\nabla u.\nabla\delta^\alpha \mathrm{d}x\right| = \left|\int_\Omega \left(\delta^{\frac{\alpha}{2}}\nabla u\right).\left(\frac{\overline{v}}{\delta^{\frac{\alpha}{2}}}\nabla\delta^\alpha\right)\mathrm{d}x\right| \leqslant \alpha\|\nabla\delta\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega;\delta^\alpha)}\|v\|_{L^2(\Omega;\delta^{-(2-\alpha)})} < \infty,$$

by Cauchy-Schwarz's inequality and (6.4).

Definition 6.1. Assumed (6.2), (6.5) and $\alpha \in (0,1)$, we say that $u \in H_0^1(\Omega;\delta^\alpha)$ is a *weak solution* of (2.1) and (1.3) in $H_0^1(\Omega;\delta^\alpha)$ if (6.3) holds and the integral condition (6.1) holds for any $v \in H_0^1(\Omega;\delta^\alpha)$.

Remark 6.2. Notice that from the Hardy's inequality (6.4) and (6.5), we get that $H_0^1(\Omega;\delta^\alpha) \hookrightarrow L^2(\Omega)$. Moreover, since

$$\delta^{-s\alpha} \in L^1(\Omega), \text{ for any } s \in (0,1), \tag{6.6}$$

we know (Drábek, Kufner and Nicolosi [11], p.30) that

$$H_0^1(\Omega;\delta^\alpha) \hookrightarrow W^{1,p_s}(\Omega), \text{ with } p_s = \frac{2s}{s+1}.$$

Remark 6.3. Obviously, there are many functions F such that $F \in L^2(\Omega;\delta^\alpha) \setminus L^2(\Omega)$ (for instance, if $F(x) \sim \frac{1}{\delta(x)^\beta}$, for some $\beta > 0$, then $F \in L^2(\Omega;\delta^\alpha)$, if $\beta < \frac{\alpha+1}{2}$ but $F \notin L^2(\Omega)$, once $\beta \geqslant \frac{1}{2}$. This fact is crucial when the nonlinear term $f(u)$ involves a singular term of the form of the Example 2.3 but with $m \in (-1,0)$ (see Díaz, Hernández and Rakotoson [9] for the real case).

Remark 6.4. We point out that in most of the papers dealing with weighted solutions of semilinear equations, the notion of solution is not justified in this way but merely by replacing the Laplace operator by a bilinear form which becomes coercive on the space $H_0^1(\Omega;\delta^\alpha)$. The second integral term in (6.1) is not mentioned $\big($since, formally, the multiplication of the equation is merely by $v \in H_0^1(\Omega;\delta^\alpha)\big)$ but then it is quite complicated to justify that such alternative weak solutions satisfy the pde equation (2.1) when they are assumed, additionally, that $\Delta u \in L^2_{\mathrm{loc}}(\Omega)$. We also mention now (although it is a completely different approach) the notion of $L^1(\Omega;\delta)$-very weak solution developed recently for many scalars semilinear equations: see, e.g., Brezis, Cazenave, Martel and Ramiandrisoa [5], Díaz and Rakotoson [10] and the references therein).

By using exactly the same *a priori* estimates, but now adapted to the space $H^1_0(\Omega;\delta^\alpha)$, we get the following result.

Theorem 6.5. *Let Ω be a nonempty bounded open subset satisfying (6.5), $V\in L^\infty(\Omega;\mathbb{R})$, $0<\alpha<1$, $0<m<1$, $(a,b,c)\in\mathbb{C}^3$ as in Theorem 2.6 and let $F\in L^2(\Omega;\delta^\alpha)$. Then we have the following result.*

1) *There exists at least one weak solution $u\in H^1_0(\Omega;\delta^\alpha)$ to (1.2) and (1.3). Furthermore, any weak solution belongs to $H^2_{\mathrm{loc}}(\Omega)$.*
2) *If, in addition, we assume the conditions of Theorem 2.12, this solution is unique in the class of $H^1_0(\Omega;\delta^\alpha)$-weak solutions.*

Remark 6.6. In the proof of the *a priori* estimates, it is useful to replace the weighted function δ by a more smooth function having the same behavior near Γ. This is the case, for instance of the first eigenfunction φ_1 of the Laplace operator,

$$\begin{cases}-\Delta\varphi_1=\lambda_1\varphi_1, & \text{in } \Omega,\\ \varphi_{1|\Gamma}=0, & \text{on } \Gamma.\end{cases}$$

It is well-known that $\varphi_1\in W^{2,\infty}(\Omega)\cap W^{1,\infty}_0(\Omega)$ and that

$$C_1\delta(x)\leqslant\varphi_1(x)\leqslant C_2\delta(x),$$

for any $x\in\Omega$, for some positive constants C_1 and C_2, independent of x. Now, with this new weighted function, it is easy to see that the second term in (6.1) does not play any important role since, for instance, when taking $v=u$ as test function, we get that

$$\begin{aligned}\mathrm{Re}\int\limits_\Omega \overline{u}\,\nabla u.\nabla\varphi_1^\alpha \mathrm{d}x&=\frac{1}{2}\int\limits_\Omega\nabla|u|^2.\nabla\varphi_1^\alpha\mathrm{d}x=-\frac{1}{2}\int\limits_\Omega|u|^2\Delta\varphi_1^\alpha\mathrm{d}x\\ &=\frac{\alpha\lambda_1}{2}\int\limits_\Omega|u|^2\varphi_1^\alpha\mathrm{d}x+\frac{\alpha(1-\alpha)}{2}\int\limits_\Omega|u|^2\varphi_1^{-(2-\alpha)}|\nabla\varphi_1|^2\mathrm{d}x\geqslant0.\end{aligned}$$

7 Some planar representations of the assumptions on the complex parameters

In this section, we give some geometric interpretation of the values of a and b. For convenience, we repeat the hypotheses (2.15) of existence and 1) of Theorem 2.12 of uniqueness. We recall that,

$$\begin{aligned}&\mathbb{A}=\mathbb{C}\setminus\mathbb{D},\\ &\mathbb{D}=\big\{z\in\mathbb{C};\mathrm{Re}(z)\leqslant0\text{ and }\mathrm{Im}(z)=0\big\}.\end{aligned}$$

For existence of weak solutions to problem (1.1) in Theorem 2.9, we suppose $(a,b)\in\mathbb{C}^2$ satisfies

$$(a,b)\in\mathbb{A}\times\mathbb{A}\quad\text{ and }\quad\begin{cases}\operatorname{Im}(a)\operatorname{Im}(b)\geqslant 0,\\ \text{or}\\ \operatorname{Im}(a)\operatorname{Im}(b)<0\ \text{ and }\ \operatorname{Re}(b)>\dfrac{\operatorname{Im}(b)}{\operatorname{Im}(a)}\operatorname{Re}(a),\end{cases}\tag{7.1}$$

while for uniqueness, we assume

$$a\neq 0,\ \operatorname{Re}(a)\geqslant 0 \text{ and } \operatorname{Re}(a\overline{b})\geqslant 0.\tag{7.2}$$

Existence. Condition (7.1) may easily be interpreted in this way: $[a,b]\cap\mathscr{D}=\emptyset$, where $\mathscr{D}$ is the geometric representation of $\mathbb{D}$, which is the half-axis of the complex plane where $\operatorname{Re}(z)\leqslant 0$. See Figures 1 and 2 below.

Uniqueness. Condition (7.2) is trivial. Indeed, we first choose $a\in\mathbb{C}\setminus\{0\}$ such that $\operatorname{Re}(a)\geqslant 0$, and we choose b with respect to a. We see a and b as vectors of $\mathbb{R}^2$. Then we write, $\overrightarrow{a}=\begin{pmatrix}\operatorname{Re}(a)\\ \operatorname{Im}(a)\end{pmatrix}$, $\overrightarrow{b}=\begin{pmatrix}\operatorname{Re}(b)\\ \operatorname{Im}(b)\end{pmatrix}$ and we have

$$\operatorname{Re}\left(a\overline{b}\right)=\operatorname{Re}(a)\operatorname{Re}(b)+\operatorname{Im}(a)\operatorname{Im}(b)=\overrightarrow{a}.\overrightarrow{b},\tag{7.3}$$

where . denotes the scalar product between two vectors of $\mathbb{R}^2$. Then the condition $\operatorname{Re}\left(a\overline{b}\right)\geqslant 0$ is equivalent to $\left|\angle(\overrightarrow{a},\overrightarrow{b})\right|\leqslant\frac{\pi}{2}\text{rad}$ (see Figure 3 below).

Remark 7.1. Thanks to (7.3), the following assertions are equivalent.

1) $(a,b)\in\mathbb{C}^2$ satisfies (7.1)–(7.2).

2) $(a,b)\in\mathbb{A}\times\mathbb{A}$ satisfies (7.2).

3) $\Big((a,b)$ satisfies (7.2)$\Big)$ and $\Big(\operatorname{Re}(a)=\operatorname{Im}(b)=0\implies\operatorname{Re}(b)>0\Big)$.

In other words, when $a\notin\mathbb{D}$, uniqueness hypothesis (7.2) implies existence hypothesis (7.1) (see Figure 4 below).

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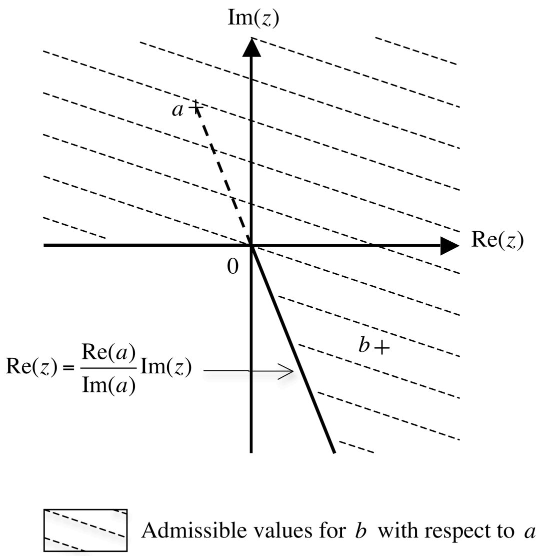

Figure 1: Existence, choice of b

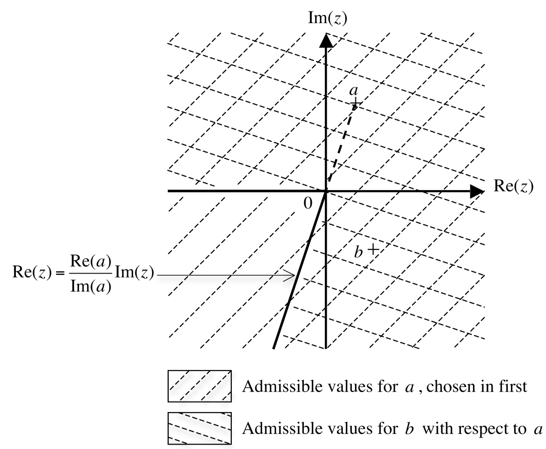

Figure 2: Existence, choice of a and b

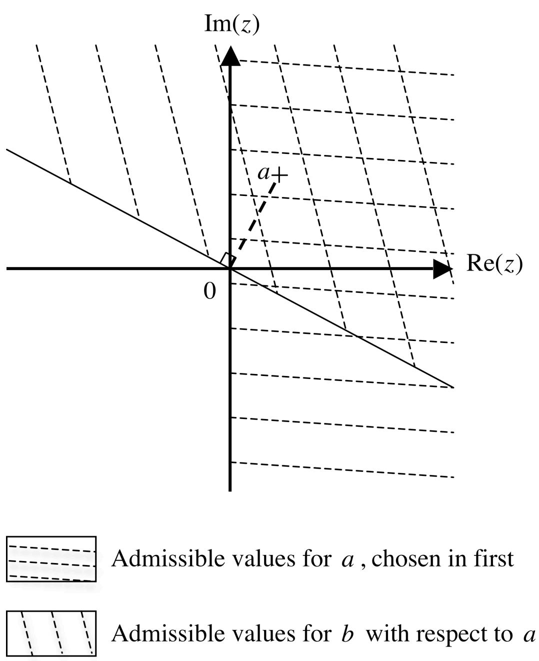

Figure 3: Uniqueness

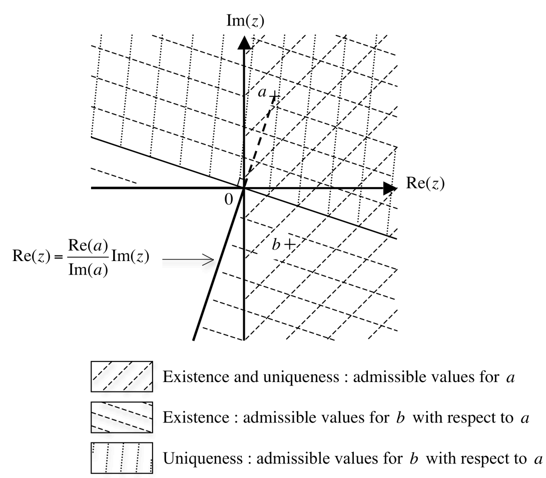

Figure 4: Uniqueness implies existence

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